Chapter 3 : Matrices (class XII)

Introduction

1) A matrix is a rectangular array of numbers, real or complex. The numbers are called the elements of the matrix or entries in the matrix.

2) It is usual to denote a matrix by a single capital A, B, C etc.

3) In condensed notation $A = [a_{ij}]$ or $A = (a_{ij})$ are used to denote the matrix A. Here a_{ij} is the element in the ith row and jth column and is sometime called the (i,j)th the element of the matrix.

4) To indicate that A is of order $m \times n$, we write $A = [a_{ij}]_{m \times n}$
$A_{m \times n}$ matrix has m rows and n column.
It has mn number of elements.

5) A general matrix of order $m \times n$ can be conveniently written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

Types of Matrices

Row matrix: Any $1 \times n$ matrix which has only one row and n columns is called a row matrix or a row vector.
e.g. $[2, 5]_{1\times 2}$ and $[1, 0, 3]_{1\times 3}$

Column matrix: Any $m \times 1$ matrix which has m rows and one column is a column matrix or a column vector.

e.g. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{2\times 1}$ and $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}_{3\times 1}$

Null or Zero matrix: The $m \times n$ matrix whose all elements are 0 is called the null matrix (or zero matrix) of the order $m \times n$. It is usually denoted by O or $O_{m\times n}$.

e.g. $\begin{bmatrix} 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square Matrix:
A matrix in which the number of rows is equal to the number of columns is called a square matrix.
A $m \times n$ matrice for which $m = n$ is called a square matrix.

Diagonal elements of a square matrix
The element a_{ij}, when $i = j$ of square matrix $A = [a_{ij}]$ are called the diagonal elements and the line along which they lie is called the principal diagonal or usually only the diagonal of the square matrix.

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$$, the diagonal elements are 1, 5, 1.

Diagonal Matrix

A square matrix in which all non-diagonal elements are zero, is called a diagonal matrix.

Example,
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Trace of a matrix

If $A = \{a_{ij}\}$ be a square matrix of order n, then the sum of its diagonal elements is defined to be the trace A.

Example, $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & -3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$, then trace of $A = 2 + (-3) + 4$
$= 3$

Scalar Matrix

A diagonal matrix in which all the diagonal elements are equal, is called a scalar matrix.

Thus the square matrix $[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} \alpha & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$ e.g. $\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$

Unit matrix or identity matrix

A square matrix each of whose diagonal element is 1 and each of whose non-diagonal element is zero is called a unit matrix.

(Or) A diagonal matrix, in which each of its diagonal element is unity, is said to be a unit matrix.

$$a_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Denoted by I_n. $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if :

(i) they are of the same order (i.e. they have same number of rows and columns)

(ii) the element in the corresponding position are equal.

Example, $[a \quad b] = [3 \quad 4]$ if and if $a=3$ $b=4$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ if and only if $a=1$, $b=2$, $c=3$ $d=4$.

Operation on Matrices

A) Addition of Matrices :

Let A and B be two matrices of the same order $m \times n$. Then their sum to be denoted by $(A+B)$ is defined to be the matrix of the order $m \times n$ obtained by adding corresponding elements of A & B.

$A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{m\times n}$

$$A + B = [a_{ij} + B_{ij}]_{m\times n}.$$

Example, $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 7 & -8 & 9 \\ 2 & 8 & -4 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1+7 & 2+(-8) & -3+9 \\ 4+2 & -5+8 & 6+(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 & 6 \\ 6 & 3 & 2 \end{bmatrix}$$

Addition is defined only for those matrices, which are of the same type. If two matrices are not of the same order, we cannot their sum.

Properties of Matrix addition :

1) Matrix addition is Commutative : $A+B = B+A$.

2) Matrix addition is Associative : $(A+B)+C = A+(B+C)$

3) Existence of additive identity : $A+O = O+A = A$
where O is null matrix.
The zero matrix O is the additive identity, which is unique.

(4) Existence of additive Inverse :
The the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m\times n}$ and is denoted by $(-A)$.

The matrix $(-A)$ is the additive inverse of the matrix A. $(-A)+A = O = A+(-A)$.

5) Difference of Matrices : $A-B = A+(-B)$.

Note : If two matrices are of the same order then only they can be added or subtracted, and they are then said to be conformable for addition or subtraction.

Example - 1 : Find the additive inverse of matrices $A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 8 \\ 3 & 2 & -1 \end{bmatrix}$

Example - 2 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$

verify that : $A+(B+C) = (A+B)+C$.

Scaler Multiplication

Let A be any $m \times n$ matrix and k be any scalar (i.e. an element of the number system k).

The $m \times n$ matrix obtained by multiplying each element of the matrix A by scaler k is said to be the scalar multiple of A by k and is denoted by kA or Ak.

$$A = [a_{ij}]_{m \times n} \text{ then } kA = [ka_{ij}]_{m \times n}.$$

Properties of Multiplication of a Matrix by a Scaler :

i) If A and B are two matrices of the same order $m \times n$, then $k(A+B) = kA + kB$, where k is any scaler.

(ii) If A is any matrix of the type $m \times n$, then $(k_1 + k_2)A = k_1 A + k_2 A$ where k_1, k_2 are scalars.

(iii) If k_1 and k_2 are two scalars and A is any $m \times n$ matrix then $k_1(k_2 A) = (k_1 k_2)A$.

(iv) If A is any $m \times n$ matrix, then $(-k)A = -(kA) = k(-A)$.

Example: $A = \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix}$

verify that $3(A+B) = 3A + 3B$.

Solved Examples

1) If $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & -17 \end{bmatrix}$

(i) What is the order of the matrix.

(ii) Write the number of elements in A

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}$

2) If a matrix has 24 elements, what are the possible orders it can have? What if it has 13 elements?

3) In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2-y \\ 0 & 5 & -\frac{2}{5} \end{bmatrix}$ write,

(i) the order of matrix

ii) the number of element in A

(iii) a_{23}, a_{31}, a_{12}.

4) Construct a 2×2 matrix where $a_{ij} = |-2i+3j|$

5) Construct a 3×4 matrix, whose elements are given by $a_{ij} = \frac{1}{2}|-3i+j|$.

6) If $\begin{bmatrix} x-y & 2y \\ 2y+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$, then write the value of $(x+y+z)$. [10]

7) Find the value of $(x+y)$ from the following equation

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 4 \end{bmatrix}$$

[6]

8) Simplify $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9) Find the values of a, b, c, d from the following equation.

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$(a=1,\ b=2,\ c=3\ d=4)$

10) Find the matrix X such that $2A+B+X=0$ where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$

Multiplication of Matrices

Let $A = [a_{ij}]_{m\times n}$ $B = [b_{jk}]_{n\times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B. Then $m\times p$ matrix $C = [c_{ik}]_{m\times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij}\cdot b_{jk}$ is called the product $C = AB$.

Example 1 : $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}_{3\times 1}$ $B = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}_{1\times 3}$ find AB & BA.

$$AB = \begin{bmatrix} 3\times 1 & 3\times 0 & 3\times 4 \\ 5\times 1 & 5\times 0 & 5\times 4 \\ 2\times 1 & 2\times 0 & 2\times 4 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 3 & 0 & 12 \\ 5 & 0 & 20 \\ 2 & 0 & 8 \end{bmatrix}_{3\times 3}$$

$$BA = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}_{1\times 3} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1\times 3 + 0\times 5 + 4\times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 + 0 + 8 \end{bmatrix}$$
$$= \begin{bmatrix} 11 \end{bmatrix}_{1\times 1}$$

Note : The product AB of two matrices A and B is defined if and only if the number of column of A = number of rows in B.

Example 2 : $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}_{2\times 3}$ find AB.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2\times 3 + 1\times 1 & 2\times 0 + 1\times(-2) & 2\times 2 + 1\times 3 \\ 1\times 3 + 0\times 1 & 1\times 0 + 0\times(-2) & 1\times 2 + 0\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & 7 \\ 3 & 0 & 2 \end{bmatrix}$$

Properties of Matrix Multiplication

1) The multiplication of matrices is usually not commutative.

a) If AB is defined, then BA may not be defined.

b) If matrix AB and matrix BA both exist, it is not necessary that they are matrices of the same order.

c) If matrix AB and matrix BA both exist and are matrices of the same order, it is not necessary that AB = BA.

d) It however does not imply that matrix AB is never equal to matrix BA.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad AB \neq BA.$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 10-22+9 & -4+10-5 & -1+0+1 \\ 30-44+18 & -12+20-10 & -3+0+2 \\ 10-33+18 & -4+15-10 & -1+0+2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & 0 \\ 4 & -2 & -1 \\ -5 & 1 & 1 \end{bmatrix}$$

Here $\underline{AB = BA}$

(ii) Matrix multiplication is Associative

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

(iii) Matrix Multiplication is distributive over addition.

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$(A + B) \cdot C = A \cdot C + B \cdot C$$

(iv) If A is a $m \times n$ matrix and I is the $n \times n$ identity matrix, then $AI = A = IA$

Positive Integral Powers of Matrices

If A is a square matrix, then AA is defined.

(i) $AA = A^2$ (ii) $A^{n+1} = A^n \cdot A$ where $n \in N$.

$$A^m A^n = A^{m+n} = A^n A^m$$

A square matrix is said to be idempotent if $A^2 = A$

A square matrix A is said to involutory if $A^2 = I$

Solved Example

1) $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. Find (i) AB (ii) BA

$\left[\text{(i)} \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix} \right.$

$\text{(ii)} \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

2) If $A = \begin{bmatrix} 3 & 2 \\ 12 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix}$. Show that $AB = 0$.

3) If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Show that $AB \neq BA$.

4) Compute the indicated products:

(i) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$

5) Solve for x: $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$ $(x = 2)$.

6) If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bC)^3 = a^3 I + 3a^2 b C$.

7) If $A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$, find $-A^2 + 6A$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

8) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ prove that $A^3 - 6A^2 + 7A + 2I = 0$.

9) (i) If $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ and $f(x) = 2x^3 + x^2 - 3x$, find $f(A)$. $\begin{bmatrix} -17 & 6 \\ -9 & -17 \end{bmatrix}$

(ii) Let $f(x) = x^2 - 5x + 6$. Find $f(A)$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

10) If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Show that $f(x) f(y) = f(x+y)$.

Use of Mathematical Induction

If the results to be proved involve positive integer n, we can use mathematical induction in proving such results.

Method : Step I. Verify the result for $n = 1$

Step II. Prove that the result is true for $n = k+1$ when it is true for $n = k$.

Example 1 : Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $(aI + bA)^n = a^n I + n a^{n-1} bA$.

Sol^n: We shall prove the result by mathematical induction. To start the induction we see that result is true for $n = 1$.

$$(aI + bA)^1 = aI + bA = aI + 1\, a^{1-1} \cdot bA$$

Now suppose that the result is true for $n = k$

$$(aI + bA)^k = a^k I + k a^{k-1} bA$$

Then $$(aI + bA)^{k+1} = (aI + bA)^k (aI + bA)$$

$$= (aI + ka^{k-1} bA)(aI + bA)$$

[$\because$ By assumption, the result is true for $n = k$]

$$= a^{k+1} I + a^k bA + ka^k bAI + ka^{k-1} \cdot b^2 A^2$$

$$= a^{k+1} I + a^k bA + ka^k bA + 0$$

[$\because$ $IA = A = AI$; $A^2 = 0$]

$$= a^{k+1} I + (k+1) a^{k+1-1} bA$$

Showing the result is true for $n = k+1$ whenever it is true for $n = k$.

Example 2 : If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

Practice Problems

1) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

2) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, show that $A^5 = 16A$

3) If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ.
$[\lambda = 6]$

4) If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, show that $A^2 = 2A$ and $A^3 = 4A$.

5) If $A = \begin{bmatrix} x & y & z \end{bmatrix}$ $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find ABC.
$[ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$

6) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$, show that $A(B+C) = AB + AC$.

7) If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, find $(A - 2I)(A - 3I)$
$[0]$

8) If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that $(A+B)(A-B) \neq A^2 - B^2$

9) If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, verify that $(A+B)^2 \neq A^2 + 2AB + B^2$.

10) Find the value of x such that $\begin{bmatrix} 1 & 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$
$[-2]$

11) If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2 + 2AB$
Find a and b.
$\begin{bmatrix} a = -1 \\ b = -2 \end{bmatrix}$

12) If $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & -2 \\ 3 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 8x + 3y & 6z & 32 \\ 4 & 12 & 26x - 5y \end{bmatrix}$
Find x, y and z.
$[x = 1, y = 3, z = 4]$

13) If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix of order 2, show that $(I+A) = (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

14) Let $f(x) = x^2 - 5x + 6$. Find $f(A)$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

15) If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$

AN 14. $f(A) = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ AN. 15. $X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$

16) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find k so that $A^2 = 8A + kI$

[$k = -7$]

17) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, where n is a positive integer.

18) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = 0$.

[$x = 9$, $y = 14$]

19) Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Show by mathematical induction that

$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ for every positive integer.

20) A trust fund has Rs 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and second bond pays 7% interest per year. Using matrix multiplication determine how to divide Rs. 30,000 among the two types of bonds, if the trusts fund must obtain an annual total interest of (i) Rs 1800 (ii) Rs. 2000

(i) Rs. 15000, Rs. 15000

(ii) Rs. 5000, 25000

Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the matrix $n \times m$ obtained from A by changing its rows into columns or columns into rows is called the transpose of matrix A and is denoted by the symbol A' or A^T.

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

The transpose of $m \times n$ matrix is a $n \times m$ matrix.

Properties of Transpose :

(i) The transpose of the transpose of a matrix is the matrix itself i.e. if A is a matrix of order $m \times n$, then $(A')' = A$.

(ii) If A is a matrix of order $m \times n$ and k is a scalar, then $(kA)' = kA'$

(iii) If A and B are two matrices conformable for addition then $(A+B)' = A' + B'$

(iv) If A and B are two matrices such that AB is defined then $(AB)' = B'A'$. This also called reversal law

Symmetric and skew-symmetric Matrices

(a) Symmetric Matrix :

A square matrix $A = [a_{ij}]$ is said to be symmetric if $(i, j)^{th}$ element is the same as its $(j, i)^{th}$ element.

Example, $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{bmatrix}$

Theorem : The necessary and sufficient condition for the matrix A to be symmetric is that $A' = A$.

(b) Skew - Symmetric Matrix

A matrix A is skew - symmetric if $A' = -A$ i.e. if $a_{ji} = -a_{ij}$ for all i and j.

Example,
$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Theorem: The necessary and sufficient condition for the matrix A to be skew - symmetric is that $A' = -A$.

Note: The symmetry of a matrix is that it is symmetric along the main diagonal, whereas the matrix is skew- symmetric if all the elements on the main diagonal are zero and the elements symmetric about the main diagonal are equal in magnitude but of opposite signs.

Some Important Properties:

(i) A matrix which is both symmetric as well as skew - symmetric is a zero matrix.

(ii) If A, B are symmetric (skew - symmetric) matrices, then so $(A+B)$

(iii) If A is a symmetric (skew - symmetric) matix, then kA is also symmetric (skew - symmetric) matrix.

(iv) If A be any square matrix then $(A+A')$ is symmetric and $(A-A')$ is skew symmetric matrix.

(v) If A and B are symmetric matrices then AB is symmetric if and only if A and B commute i.e. $AB = BA$

(vi) If A be any square matrix, then AA' and $A'A$ are both symmetric matrices.

(vii) If A and B are symmetric matrices then $AB - BA$ is skew-symmetric matrix.

(viii) The matrix $B'AB$ is symmetric or skew-symmetric as A is symmetric or skew-symmetric.

(ix) Every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Solved Examples

1) (i) Show that $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & -1 \\ 5 & -1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

2) If $A = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}$, show that $A + A'$ is a symmetric matrix.

3) If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that AA' and $A'A$ are both symmetric.

4) If $A = \begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$, show that $\frac{1}{2}(A - A')$ is a skew-symmetric matrix.

5) Express the matrix $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix.

6) If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, find the value of x. [$x = -4$]

7) Write the symmetric part of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix}$

[Hint. Symmetric part of $A = \frac{1}{2}(A + A')$].

8) Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric. Find a and b. [$a = \frac{-2}{3}$ $b = \frac{3}{2}$]

9) If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix. Find a, b and c [$a = -2, b = 0, c = -3$]

10) Express the matrix $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

11) Express $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Elementry operations (Transformations).

There are six operations (transformations) of a matrix - three for rows and three for columns, which are known as elementary operation or elementry transformations.

(I) Interchange of any two rows (columns)

$R_i \leftrightarrow R_j$ $\quad C_i \leftrightarrow C_j$

(II) Multiplication of the elements of any (row) column by a non-zero number. $R_i \to k R_i$ $\quad C_i \to k C_i$

(III) Addition to the elements any row (column) the corresponding elements of any other row (column) multiplied by any non-zero number.

$R_i \to R_i + kR_j$ $\quad C_i \to C_i + kC_j$

Equivalent Matrices

Two matrices are said to be equivalent if one can be obtained from the other by a sequence of elementry operations. We write $A \sim B$.

Example,
$$\begin{bmatrix} 5 & 6 & 7 \\ -1 & \sqrt{3} & 1 \\ 2 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \\ 2 & 4 & 8 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

Invertible Matrix and Inverse of a Matrix

Any n-rowed square matrix A is said to be invertible if there exists an n-rowed B matrix such that

$AB = BA = I_n$ $\quad$ (I_n - unit matrix of order n)

B is called the inverse of A or reciprocal of A.

Inverse of A is denoted by A^{-1}.

Solved Examples

1) Using elementary transformation, find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, if it exist.

$\left[A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \right]$

2) Using elementary row operations, find the inverse of the following matrix $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$\left[A^{-1} = \begin{matrix} 3 & -5 \\ -1 & 2 \end{matrix} \right]$

3) Find the inverse of $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ using elementary row transformation.

4) Using elementary row operations find A^{-1}. $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}$.

Ans. 3) $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Ans 4) $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$