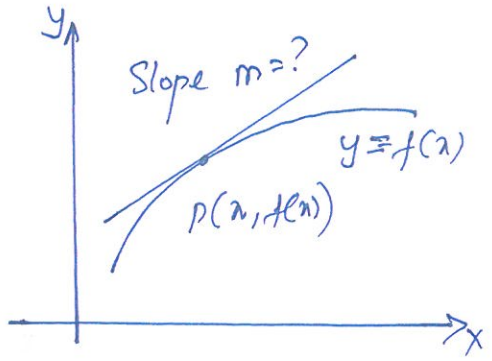
Chapter 7 : INTEGRALS (XII)

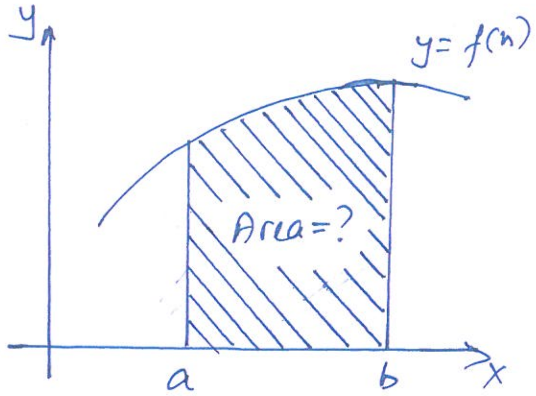
Introduction

In mathematics (in calculus), the derivative is a way to show instantaneous rate of change : that is, the amount by which a function is changing at one given point. For functions, that act on real numbers, it is the slope of the tangent line at a given point on a graph.

The second fundamental problem addressed by calculus is the problem of areas, that is, the problem of determining the area of a region of the plane bounded by various curves. Finding an area is equivalent to finding an antiderivatives, or as we prefer to say, finding an integral.



The tangent-line problem motivates differential calculus



The area problem motivates integral calculus

The relationship between area and antiderivatives is called the fundamental theorem of calculus. It provides a vital connection between the operations of differentiation and integration.

Fundamental Theorem of Calculus connects Indefinite and Definite Integrals. This makes the Definite integral as practical both for Science and Engineering.

Indefinite Integrals

Integration is the inverse process of Differentiation. Here we are given the derivative of a function and we are to find its primitive i.e. the original function. Such process is known as integration or anti-differentiation or anti-derivatives.

So we introduce a new symbol viz. $\int f(x)\,dx$, which represents the entire class of anti-derivatives and is read as "indefinite integral of 'f' with respect to x."

Symbolically : $\int f(x)\,dx = F(x) + C$

Given : $\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\,dx$.

For example, consider $f(x) = x^3$ on $[-2, 3]$

$$F'(x) = \frac{d}{dx}(x^3) = 3x^2 = f(x) \text{ for all } x \text{ in } (-2, 3)$$

$\therefore$ by def. $F(x) = x^3$ is anti-derivative of $f(x) = 3x^2$ on $[-2, 3]$.

Constant of integration

We know that (i) $\frac{d}{dx}(x^3) = 3x^2$ (ii) $\frac{d}{dx}(x^3+2) = 3x^2$

(iii) $\frac{d}{dx}(x^3-7) = 3x^2$ (iv) $\frac{d}{dx}(x^3+c) = 3x^2$

$\therefore$ integral of $3x^2$ may be x^3, x^3+2, x^3-7 or x^3+C, where c is any arbitrary constant.

Thus if $F(x)$ is any any integral of $f(x)$, $F(x)+C$ is also an integral of $f(x)$, where c is any arbitrary constant.

Comparison between differentiation and Integration

1. ALL THE FUNCTIONS ARE NOT DIFFERENTIABLE.	1. ALL THE FUNCTIONS ARE NOT INTEGRABLE.
2. DERIVATIVE OF A FUNCTION IS UNIQUE.	2. INTEGRAL OF A FUNCTION IS NOT UNIQUE.
3. WHEN POLYNOMIAL IS DIFFERENTIATED WE GET A POLYNOMIAL OF DEGREE 1 LESS THAN IT.	3. WHEN POLYNOMIAL IS INTEGRATED WE GET A POLYNOMIAL OF DEGREE 1 MORE THAN IT.
4. IT IS USED TO FIND VELOCITY WHEN DISPLACEMENT IS GIVEN	4. IT IS USED TO FIND VELOCITY WHEN ACCLERATION IS GIVEN, centre of mass, momentum etc.
5. The derivative has a geometric meaning viz. the slope of the tangent to a curve.	5. The integral has a geometric meaning viz. the area of some region.

6. Both are operations on functions. Each gives function as answer.

7. Both are linear

8. Differentiation and integration are inverse of each other.

9. Both satisfy the following properties of linearity:

(i) $\frac{d}{dx}[k_1 f_1(x) + k_2 f_2(x)] = k_1 \frac{d f_1(x)}{dx} + k_2 \frac{d f_2(x)}{dx}$

(ii) $\int [k_1 f_1(x) + k_2 f_2(x)]dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx$

where k_1 and k_2 are constants.

Fundamental formulae or Standard forms of integrals.

Since	Therefore
1) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$ In particular $\frac{d(x)}{dx} = 1 \cdot x^0 = 1$	1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (if $n \neq -1$) In particular $\int 1 \cdot dx = \int x^0 dx = x + c$ Note: The power rule fails when $n = -1$.
2) $\frac{d}{dx}\left[\frac{(ax+b)^{n+1}}{a(n+1)}\right] = (ax+b)^n$	2) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ (if $n \neq -1$)
3) $\frac{d}{dx}(\log\|x\|) = \frac{1}{x}$	3) $\int \frac{1}{x} dx = \log\|x\| + c$
4) $\frac{d}{dx}\left(\frac{\log\|ax+b\|}{a}\right) = \frac{1}{ax+b}$	4) $\int \frac{dx}{ax+b} = \frac{1}{a}\log\|ax+b\| + c$
5) $\frac{d(e^x)}{dx} = e^x$	5) $\int e^x dx = e^x + c$
6) $\frac{d}{dx}(e^{ax+b}) = a e^{ax+b}$	6) $\int e^{ax+b} dx = \frac{e^{ax+b}}{a}$
7) $\frac{d(a^x)}{dx} = a^x \log a$	7) $\int a^x dx = \frac{a^x}{\log a} + c$
8) $\frac{d(\sin x)}{dx} = \cos x$	8) $\int \cos x \, dx = \sin x + c$
9) $\frac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)$	9) $\int \cos(ax+b) dx = \frac{1}{a}\sin(ax+b) + c$
10) $\frac{d(\cos x)}{dx} = -\sin x$	10) $\int \sin x \, dx = -\cos x + c$

Note: If we know the integral of $f(x)$, then the integral of $f(ax+b)$ is obtained by replacing x by $ax+b$ and dividing the result by a.

Since	Therefore
11) $\frac{d[\cos(ax+b)]}{dx} = -a\sin(ax+b)$	11) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$
12) $\frac{d(\tan x)}{dx} = \sec^2 x$	12) $\int \sec^2 x\,dx = \tan x + C$
13) $\frac{d[\tan(ax+b)]}{dx} = a\sec^2(ax+b)$	13) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$
14) $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$	14) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$
15) $\frac{d[\cot(ax+b)]}{dx} = -a\operatorname{cosec}^2(ax+b)$	15) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$
16) $\frac{d(\sec x)}{dx} = \sec x \cdot \tan x$	16) $\int \sec x \cdot \tan x = \sec x + C$
17) $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cdot \cot x$	17) $\int \operatorname{cosec} x \cdot \cot x = -\operatorname{cosec} x + C$
18) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$	18) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
19) $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$	19) $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$
20) $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$	20) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
21) $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$	21) $\int \frac{-1}{1+x^2}\,dx = \cot^{-1} x + C$
22) $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	22) $\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$
23) $\frac{d(\operatorname{cosec}^{-1} x)}{dx} = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	23) $\int \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1} x + C$

Solved Examples

1) Write down the integral of (i) x^2 (ii) $\sqrt{x}$ (iii) 1

Sol^n (i) $\int x^2 dx = \dfrac{x^{2+1}}{2+1} + C = \frac{1}{3}x^3 + C$

(ii) $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3} x^{2/3} + C$

(iii) $\int 1 \cdot dx = \int x^0 dx = \dfrac{x^{0+1}}{0+1} + C = x + C$.

2) Evaluate $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

Sol^n: $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x} \cdot \frac{1}{\sqrt{x}} = x + \frac{1}{x} - 2$

$\therefore \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int \left(x + \frac{1}{x} - 2\right) dx = \int x \cancel{dx} + \int \frac{1}{x} dx - 2\int dx$

$= \dfrac{x^2}{2} + \log|x| - 2x + C$.

3) Evaluate : (i) $\int \sqrt{1+\cos x}\, dx$ (ii) $\int \dfrac{1+\sin x}{1-\sin x} dx$ iii $\int \sqrt{1-\sin x}\, dx$

Sol^n (i) $1 + \cos x = 2\cos^2 \frac{x}{2}$

$\therefore \int \sqrt{1+\cos x}\, dx = \int \sqrt{2\cos^2 \frac{x}{2}}\, dx = \sqrt{2} \int \cos \frac{x}{2}\, dx$

$= \sqrt{2}\, \dfrac{\sin \frac{x}{2}}{\frac{1}{2}} = 2\sqrt{2} \sin \frac{x}{2} + C$.

(ii) $\frac{1+\sin x}{1-\sin x} = \frac{(1+\sin x)}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1^2 - \sin^2 x}$

$= \frac{1+\sin^2 x + 2\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x}$

$= \sec^2 x + \tan^2 x + 2\sec x \cdot \tan x$

$\therefore \int \frac{1+\sin x}{1-\sin x} dx = \int (\sec^2 x + \tan^2 x + 2\sec x \cdot \tan x)\, dx$

$= \int \sec^2 x\, dx + 2\int \sec x \cdot \tan x\, dx + \int \tan^2 x\, dx$

$= \int \sec^2 x\, dx = \int (\sec^2 x - 1)\, dx + 2\int \sec x \cdot \tan x \cdot dx$

$= 2\int \sec^2 x\, dx - \int dx + 2\int \sec x \cdot \tan x\, dx$

$= 2\tan x - x + \sec x + C$

(iii) $1 - \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}$

$= \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$

$\therefore \int \sqrt{1-\sin x} = \int \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2} dx = \int \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) dx$

$= \frac{\sin\frac{x}{2}}{1/2} - \frac{-\cos\frac{x}{2}}{1/2} + C$

$= 2\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right] + C.$

4) Find the following integrals:

(i) $\int (\tan x + \cot x)^2 dx$ (ii) $\int \frac{\cos 4x}{\sin^2 x} dx$

ANS: $\tan x - \cot x + C$ ANS: $-\cot x - 4x - 2\sin 2x + C$.

3) Evaluate the following: (i) $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right) dx$

(ii) $\int \sin^{-1}(\cos x)dx$ (iii) $\int \tan^{-1}(\sec x + \tan x)dx$

ANS (i) $2\tan x + 2\sec x - x + C$ (ii) $\frac{x^2}{2} + C$ (iii) $\frac{\pi x}{2} - \frac{x^2}{2} + C$.

4) Find the following integrals: $\int \sin^4 x\, dx$ ANS: $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

5) Find the following integrals: $\int \sin x \sin 2x \sin 3x\, dx$

ANS: $\frac{1}{4}\left(-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right) + C$.

6) Find the integral: $\int \frac{5\cos^3 x + 7\sin^3 x}{3\sin^2 x \cdot \cos^2 x} dx$ ANS: $-\frac{5}{3}\operatorname{cosec} x + \frac{7}{3}\sec x + C$

7) Evaluate: $\int \sin^4 x \cos^4 x\, dx$ ANS: $\frac{1}{64}\left[\frac{3}{2}x + \frac{1}{16}\sin 8x - \frac{1}{2}\sin 4x\right] + C$.

8) Evaluate: $\int \cos^4 2x\, dx$ ANS: $\frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C$.

9) Evaluate: $\int \sin^2(2x+5)\, dx$ ANS: $\frac{x}{2} - \frac{\sin(4x+10)}{8} + C$.

10) Evaluate: $\int \cos 2x \cdot \cos 4x \cdot \cos 6x\, dx$

ANS: $\frac{1}{4}\left(x + \frac{1}{12}\sin 12x + \frac{1}{8}\sin 8x + \frac{1}{4}\sin 4x\right) + C$.

MISCELLANEOUS EXAMPLES

1) Evaluate : $\int (e^{a\log x} + e^{x\log a})\, dx$

Solⁿ: Since $e^{\log f(x)} = f(x)$

$$e^{a\log x} = e^{\log x^a} = x^a$$

$$e^{x\log a} = e^{\log a^x} = a^x$$

$$\int (e^{a\log x} + e^{x\log a})\, dx = \int (x^a + a^x)\, dx$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + C \qquad \left[\because \int a^x dx = \frac{a^x}{\log a} + c\right]$$

2) Evaluate : $\int \frac{1+x^5}{1+x} dx$

Solⁿ: $\frac{1+x^5}{1+x} = \frac{1-(-x)^5}{1-(-x)}$

As $\frac{a(1-r^n)}{1-r}$ is a sum of a G.P. whose first term is 'a', common ration 'r' and number of terms is n.

$\therefore \frac{1-(-x)^5}{1-(-x)}$ is a sum of a G.P. whose first term is 1 common ratio is $-x$') and number of terms is 5.

$$I = \int \frac{1+x^5}{1+x} dx = \int (1 - x + x^2 - x^3 + x^4)\, dx$$

$[\because a = 1,\ ar = -x,\ ar^2 = x^2,\ ar^3 = -x^3 \text{ and } ar^4 = x^4]$

$$I = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + c.$$

3) Evaluate : $\int \frac{x^4+1}{x^2+1}\,dx$

Sol^n : $\int \frac{x^4+1}{x^2+1}\,dx = \int \frac{(x^4-1)+2}{1+x^2}\,dx = \int \frac{x^4-1}{x^2+1}\,dx + \int \frac{2}{1+x^2}\,dx$

$$= \int \frac{\cancel{(x^2+1)}(x^2-1)}{\cancel{(x^2+1)}}\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$= \int x^2\,dx - \int dx + 2\tan^{-1}x + C$$

$$I = \frac{x^3}{3} - x + 2\tan^{-1}x + C$$

4) Evaluate : $\int \frac{dx}{\sin^2 x \cos^2 x}$

Sol^n : $I = \int \frac{dx}{\sin^2 x\cdot\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cdot\cos^2 x}\,dx$

$$= \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cdot\cos^2 x}\,dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x\cdot\cancel{\cos^2 x}}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{dx}{\sin^2 x}$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx$$

$$= \tan x + (-\cot x) + C$$

$$= \tan x - \cot x + C$$

5) Evaluate : $\int \frac{1+\cos x}{1-\cos x}\,dx$

Sol^n : $I = \int \frac{\cancel{2}\cos^2\frac{x}{2}}{\cancel{2}\sin^2\frac{x}{2}}\,dx = \int \cot^2\frac{x}{2}\,dx = \int \left(\operatorname{cosec}^2\frac{x}{2} - 1\right)dx$

$$= \frac{-\cot\frac{x}{2}}{1/2} - x + C = -2\cot\frac{x}{2} - x + C.$$

6) Evaluate : $\int \left(\frac{1+\sin x}{1-\sin x}\right) dx$.

Sol^n: $\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x}$

$= \frac{1+\sin^2 x + 2\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x}$

$= \sec^2 x + \tan^2 x + 2\sec x \cdot \tan x$

$\int \frac{1+\sin x}{1-\sin x} dx = \int \sec^2 x\, dx + \int \tan^2 x\, dx + 2\int \sec x \cdot \tan x\, dx$

$= \int \sec^2 x\, dx + \int (\sec^2 x - 1)\, dx + 2\int \sec x \cdot \tan x\, dx$

$= 2\int \sec^2 x\, dx - \int dx + 2\int \sec x \cdot \tan x\, dx$

$= 2\tan x - x + 2\sec x + C$

7) Evaluate: $\int \cos x \cdot \cos 2x \cdot \cos 3x \cdot dx$

Sol^n: Using $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$\therefore 2\cos x \cdot \cos 3x = \cos 4x + \cos 2x$

$\therefore I = \int \frac{1}{2}(2\cos x \cdot \cos 3x)\cos 2x\, dx$

$= \frac{1}{2}\int (\cos 4x + \cos 2x)\cos 2x\, dx$

$= \frac{1}{2}\int \cos 4x \cdot \cos 2x\, dx + \frac{1}{2}\int \cos^2 2x\, dx$

$= \frac{1}{4}\int (2\cos 4x \cdot \cos 2x)dx + \frac{1}{2\cdot 2}\int (1+\cos 4x)dx$

$= \frac{1}{4}\int (\cos 6x + \cos 2x)dx + \frac{1}{4}\int dx + \frac{1}{4}\int \cos 4x\, dx$

$$I = \frac{1}{4}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + \frac{x}{4} + \frac{\sin 4x}{4\times 4} + C$$

$$= \frac{\sin 6x}{24} + \frac{\sin 2x}{8} + \frac{x}{4} + \frac{\sin 4x}{16} + C$$

8) Evaluate: $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\, dx \quad \left(0 < x < \frac{\pi}{2}\right)$

Sol^n: $1-\sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$$1-\sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$$

$$\therefore\ 1+\sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$$

$$\sqrt{\frac{1-\sin x}{1+\sin x}} = \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} = \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$\left[\because \frac{1-\tan\theta}{1+\tan\theta} = \tan\left(\frac{\pi}{4}-\theta\right) \text{ also } \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$I = \int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\, dx = \int \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] dx$$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2}\right) dx = \frac{\pi}{4}\int dx - \frac{1}{2}\int x\, dx$$

$$= \frac{\pi}{4}x - \frac{1}{2}\frac{x^2}{2} + C$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + C.$$

9) Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

Sol^n: We know $a^3+b^3 = (a+b)^3 - 3ab(a+b)$

$$(\sin^2 x)^3 + (\cos^2 x)^3 = (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cdot \cos^2 x(\sin^2 x + \cos^2 x)$$

$$= (1)^3 - 3\sin^2 x \cdot \cos^2 x \, (1)$$

$$= 1 - 3\sin^2 x \cdot \cos^2 x$$

$$\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} = \frac{1 - 3\sin^2 \cos^2 x}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x \cdot \cos^2 x} - 3$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} - 3 = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} - 3$$

$$= \sec^2 x + \operatorname{cosec}^2 x - 3$$

$$\therefore \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx = \int \sec^2 x + \int \operatorname{cosec}^2 x - 3\int dx$$

$$I = \tan x - \cot x - 3x + C$$

10) Evaluate : $\int 4 \cos\left\{2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right\} dx$

put $x = \cos\theta \quad \theta = \cos^{-1} x$

$$\frac{1+x}{1-x} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{\cancel{2}\cos^2\frac{\theta}{2}}{\cancel{2}\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2}$$

$$\sqrt{\frac{1+x}{1-x}} = \cot\frac{\theta}{2}$$

$$\therefore \cot^{-1}\sqrt{\frac{1+x}{1-x}} = \cot^{-1}\left(\cot\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1} x$$

$$I = \int 4 \cos\left(\cancel{2} \cdot \frac{1}{\cancel{2}} \cos^{-1} x\right) dx$$

$$= \int 4 \cos\left(\cos^{-1} x\right) dx$$

$$= \int 4x \, dx$$

$$= 4 \frac{x^2}{2} = 2x^2 + C.$$

11) Evaluate : $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx$

$$\frac{1 + \cos 4x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}} = \frac{2\cos^2 2x}{\cos^2 x - \sin^2 x} \times \sin x \cdot \cos x$$

$$= \frac{2 \cos^{\cancel{2}} 2x \times \sin x \cdot \cos x}{\cancel{\cos 2x}} = \frac{1}{2} 2(\cos 2x) \cdot 2 \sin x \cdot \cos x$$

$$= \frac{1}{2} (2\cos 2x)(\sin 2x) = \frac{1}{2} (2 \cos 2x \cdot \sin 2x)$$

$$= \frac{1}{2} \sin 4x$$

$$\therefore \int \frac{\cos 4x + 1}{\cot x - \tan x} dx = \frac{1}{2} \int \sin 4x \, dx = -\frac{1}{8} \cos 4x + C.$$

Techniques of Integration

Many times the function to be integrated looks quite simple, but does not seem to fit in a standard form. In such cases, we have to adopt other techniques in finding the integrals :

(1) Integration by substitution
(2) Integration by parts.
(3) Integration by resolution into partial fraction.

Integration by Substitution

Here we aim at reducing the given integral to simpler one by the change of variable involved to a new one by means of a suitable substitution.

Example: Evaluate $\int \frac{2x}{1+x^2}\,dx$

Put $1+x^2 = t$ then $2x\,dx = dt$

then $I = \int \frac{dt}{t} = \log|t| + C = \log|1+x^2| + C$

Example: Evaluate $\int \frac{(\log x)^2}{x}\,dx$

Put $\log x = t$ then $\frac{1}{x}\,dx = dt$

$$I = \int t^2\,dt = \frac{t^3}{3} + C$$

$$= \frac{1}{3}(\log x)^3 + C.$$

Theorem : The integral of a function whose numerator is the exact differential coefficient of the denominator is log(denominator)

i.e. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c.$

Theorem : $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ when $n \neq -1$.

Theorem : Let $f(x)$ be a continuous function and $u = g(x)$ be a function with continuous derivative $g'(x)$

then, $\int f(u)\,du = \int f[g(x)]\,g'(x)\,dx.$

Theorem : If $\int f(x)\,dx = F(x) + C$, then $\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + C$

More standard Results

(i) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\,dx = \log|\sin x| + C$

(iii) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(iv) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

Note : There is no hard and fast rule for proper suitable substitution. However, the following are useful:

(i) If the integrand contains the t-ratio of $f(x)$ or logarithmic of $f(x)$ or an exponential of $f(x)$ in which index is $f(x)$ put $f(x) = t$

(ii) If integrand is a rational function of e^x, put $e^x = t$

Solved Examples

(1) Evaluate the following : (i) $\int \frac{2x+9}{x^2+9x+30}\,dx$ (ii) $\int \frac{x^2+1}{(x+1)^2}\,dx$

Soln: (i) put $x^2+9x+30 = t$

$2x+9 = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log|t| + c = \log|x^2+9x+30| + c.$

(ii) $I = \int \frac{x^2+1}{(x+1)^2}\,dx = \int \frac{x^2+1+2x-2x}{(x+1)^2}\,dx$

$= \int \frac{(x+1)^2 - 2x}{(x+1)^2}\,dx$

$= \int dx - 2\int \frac{x}{(x+1)^2}\,dx$

$= x - 2\int \frac{x+1-1}{(x+1)^2}\,dx = x - 2\int \frac{1}{x+1}\,dx + 2\int (x+1)^{-2}\,dx$

$= x - 2\log|x+1| + \frac{2\,(x+1)^{-1}}{-1} + c$

$= x - \frac{2}{x+1} - 2\log|x+1| + c \qquad x > -1.$

2) Evaluate : $\int x\sqrt{x^2+1}\,dx$

Soln : put $x^2+1 = t \qquad 2x\,dx = dt \qquad x\,dx = \frac{dt}{2}$

$\therefore \int x\sqrt{x^2+1}\,dx = \int \frac{\sqrt{t}}{2}\,dt = \frac{1}{2}\int (t)^{1/2}\,dt$

$= \frac{1}{2}\,\frac{t^{3/2}}{3/2} + c = \frac{1}{3}\,(x^2+1)^{3/2} + c.$

3) Evaluate : $\displaystyle\int \frac{dx}{\sqrt{x+3}-\sqrt{x+2}}$

Sol^n : Rationalising the denominator

$$I = \int \frac{\sqrt{x+3}+\sqrt{x+2}}{(\sqrt{x+3})^2-(\sqrt{x+2})^2}\,dx = \int \frac{\sqrt{x+3}+\sqrt{x+2}}{x+3-x-2}\,dx$$

$$= \int (x+3)^{\frac{1}{2}}\,dx + \int (x+2)^{\frac{1}{2}}\,dx$$

$$= \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left[(x+3)^{3/2} + (x+2)^{3/2}\right] + C$$

4) Find : $\displaystyle\int \cos 6x \sqrt{1+\sin 6x}\,dx$

Sol^n : put $1+\sin 6x = t$ $\qquad 6\cos 6x\,dx = dt$

$\therefore \quad \cos 6x\,dx = \dfrac{dt}{6}$

$$I = \int \frac{(t)^{1/2}}{6}\,dt = \frac{1}{6}\int t^{1/2}\,dx = \frac{1}{6}\,\frac{t^{3/2}}{3/2} + C$$

$$= \frac{1}{9}\,t^{3/2} + C = \frac{1}{9}(1+\sin 6x)^{3/2} + C.$$

RULE: To integrate $\int \sin^m x \cdot \cos^n x \, dx$, when either m or n is a positive odd integer, whatever the other may be.

(i) If the index of $\sin x$ is a positive odd integer, put $\cos x = t$.

(ii) If the index of $\cos x$ is a positive odd integer, put $\sin x = t$.

Note: If the indices of $\sin x$ and $\cos x$ are both odd, then consider the smaller one as odd.

5) integrate : $\int \cos^3 x \cdot \sin^4 x \, dx$

Sol^n: Since the index of $\cos x$ is 3 i.e positive odd integer

$\therefore$ put $\sin x = t$

$\cos x \, dx = dt$

$$\therefore \int \cos^3 x \cdot \sin^4 x \, dx = \int \cos^2 x \cdot \cos x \cdot \sin^4 x \, dx$$

$$= \int (1-\sin^2 x) \sin^4 x \cos x \, dx$$

$$= \int (1-t^2) t^4 \, dt = \int (t^4 - t^6) \, dt$$

$$= \int t^4 dt - \int t^6 dt = \frac{t^5}{5} - \frac{t^7}{7} + c$$

$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c.$$

6) Evaluate $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}\,dx$.

Sol^n Put $a\cos^2 x + b\sin^2 x = t$

$2a\cos x(-\sin x) + 2b\sin x\cos x = dt$

$2\cos x\sin x[b-a] = dt$

$\sin 2x = \frac{dt}{b-a}$

$I = \int \frac{1}{t}\cdot\frac{dt}{(b-a)} = \frac{1}{b-a}\int \frac{1}{t}\,dt = \frac{1}{b-a}\log|t| + C$

$= \frac{1}{b-a}\log|a\cos^2 x + b\sin^2 x| + C.$

7) Evaluate : $\int \frac{dx}{1+\tan x}$

Sol^n : $\frac{1}{1+\tan x} = \frac{1}{1+\frac{\sin x}{\cos x}} = \frac{\cos x}{\sin x + \cos x}$

$= \frac{1}{2}\cdot\frac{2\cos x}{(\sin x + \cos x)} = \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2(\cos x + \sin x)}$

$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t$; $(-\sin x + \cos x)\,dx = dt$

$(\cos x - \sin x)\,dx = dt$

$I = \frac{x}{2} + \frac{1}{2}\int \frac{dt}{t} = \frac{x}{2} + \frac{1}{2}\log|t| + C$

$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C.$

8) Evaluate : (i) $\int \frac{\sin(x-a)}{\sin x}\,dx$ (ii) $\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$.

Sol^n (i) $\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$$I = \int \frac{\sin x \cos a - \cos x \cdot \sin a}{\sin x}\,dx$$

$$= \int \cos a\,dx - \sin a \int \frac{\cos x}{\sin x}\,dx$$

$$= x\cos a - \sin a \log|\sin x| + C \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C\right]$$

(ii) $$\int \frac{\sin(x-a)}{\sin(x+a)}\,dx = \int \frac{\sin[(x+a)-2a]}{\sin(x+a)}\,dx$$

$$= \int \frac{\sin(x+a)\cos 2a - \cos(x+a)\cdot\sin 2a}{\sin(x+a)}\,dx$$

$$= \int \cos 2a\,dx - \sin 2a \int \frac{\cos(x+a)}{\sin(x+a)}\,dx$$

$$= x\cos 2a - \sin 2a \log|\sin(x+a)| + C$$

9) Evaluate : $\int \frac{1}{\sin(x-a)\cdot\sin(x-b)}\,dx$

Sol^n : $$I = \frac{1}{\sin(a-b)} \cdot \int \frac{\sin(a-b)}{\sin(x-a)\cdot\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \cdot \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\cdot\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\cdot\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \left[\int \frac{\cos(x-a)}{\sin(x-a)}\,dx - \int \frac{\cos(x-b)}{\sin(x-b)}\,dx\right]$$

$$i = \frac{1}{\sin(a-b)}\left[\log|\sin(x-a)| - \log|\sin(x-b)| + c.\right].$$

$$= \operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c.$$

10) Integrate : $\int \frac{\tan^4\sqrt{x}\cdot\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

Sol^n: Put $\tan\sqrt{x} = t$ $\left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$

$\sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}\,dx = dt \Rightarrow \frac{\sec^2\sqrt{x}}{\sqrt{x}} = 2dt$

$$i = \int t^4(2\,dt) = 2\int t^4\,dt$$

$$= \frac{2}{5}\tan^5\sqrt{x} + c$$

11) Evaluate : $\int \frac{(x+1)e^x}{\sin^2(xe^x)}\,dx$

Sol^n: Put $xe^x = t \Rightarrow (xe^x + e^x\cdot 1)\,dx = dt$

$e^x(1+x)\,dx = dt$

$$\therefore\ i = \int\frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t\,dt = -\cot t + c$$

$$= -\cot(xe^x) + c.$$

12) Evaluate : $\int \frac{\sin x}{\sqrt{1+\sin x}}\,dx$

$$I = \int \frac{(1+\sin x)-1}{\sqrt{1+\sin x}}\,dx = \int \frac{1+\sin x}{\sqrt{1+\sin x}} - \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \sqrt{1+\sin x}\,dx - \int \frac{dx}{\sqrt{1+\sin x}}.$$

$$I = I_1 - I_2$$

$$I_1 = \int \sqrt{1+\sin x}\,dx = \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx$$

$$\left[\because\ 1+\sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2\right]$$

$$I_1 = -\frac{\cos\frac{x}{2}}{1/2} + \frac{\sin\frac{x}{2}}{1/2} = 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C$$

$$I_2 = \int \frac{dx}{\sqrt{1+\sin x}} = \int \frac{dx}{\sin\frac{x}{2} + \cos\frac{x}{2}}\,dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\frac{x}{2}\cdot\frac{1}{\sqrt{2}} + \cos\frac{x}{2}\cdot\frac{1}{\sqrt{2}}}\,dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\frac{x}{2}\sin\frac{\pi}{4} + \cos\frac{x}{2}\cos\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\cos\left(\frac{x}{2}-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}} \int \sec\left(\frac{x}{2}-\frac{\pi}{4}\right) dx = \frac{1}{\sqrt{2}}\, \frac{\log\left|\sec\left(\frac{x}{2}-\frac{\pi}{4}\right)+\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right|}{1/2}$$

$$= \sqrt{2}\log\left|\sec\left(\frac{x}{2}-\frac{\pi}{4}\right)+\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right|$$

$$\therefore\ I = 2\left[\sin\frac{x}{2} - \cos\frac{x}{2}\right] - \sqrt{2}\log\left|\sec\left(\frac{x}{2}-\frac{\pi}{4}\right)+\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right| + K$$

13) Evaluate : $\int \frac{dx}{a\sin x + b\cos x}$

put $a = r\sin\theta$ $b = r\cos\theta$

so that $r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\frac{a}{b}$

$$I = \int \frac{dx}{r\sin\theta\cdot\sin x + r\cos\theta\cdot\cos x} = \frac{1}{r}\int \frac{dx}{\cos(x-\theta)}$$

$$= \frac{1}{r}\int \sec(x-\theta)\,dx = \frac{1}{r}\log|\sec(x-\theta)+\tan(x-\theta)|+C.$$

where $r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\frac{a}{b}$.

14) $\int \sqrt{1+2\tan x(\tan x+\sec x)}\,dx$

Soln: $I = \int \sqrt{\sec^2 x - \tan^2 x + 2\tan^2 x + 2\tan x\cdot\sec x}\,dx$

$$= \int \sqrt{\sec^2 x + \tan^2 x + 2\tan x\cdot\sec x}\,dx$$

$$= \int \sqrt{(\sec x+\tan x)^2}\,dx$$

$$= \int (\sec x+\tan x)\,dx = \int \sec x\,dx + \int \tan x\,dx$$

$$= \log|\sec x+\tan x| + \log|\sec x| + C.$$

MISCELLANEOUS EXAMPLES

1) Evaluate : $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$

Sol^n: Put $\frac{1}{x} = t$ then $-\frac{1}{x^2} dx = dt \Rightarrow \frac{1}{x^2} dx = -dt$

$$I = \int \cos t\,(-dt) = \int -\cos t\, dt = -\sin t + C$$

$$= -\sin\left(\frac{1}{x}\right) + C$$

2) Evaluate : $\int \frac{e^x+1}{e^x-1} dx$

$$I = \int \frac{e^x - 1 + 2}{e^x - 1} dx = \int dx - \int \frac{2}{e^x - 1} dx$$

$$= x - 2\int \frac{1/e^x}{\frac{e^x}{e^x} - \frac{1}{e^x}} dx = x - \int \frac{e^{-x}}{1 - e^{-x}} dx$$

put $1 - e^{-x} = t \Rightarrow -e^{-x} dx = dt$

$e^{-x} dx = -dt$

$$\therefore I = x - \int \frac{-dt}{t} = x + \log|t| + C$$

$$I = x + \log|1 - e^{-x}| + C.$$

3) Write the value of $\int \frac{1 + \cot x}{x + \log(\sin x)} dx$

Sol^n: put $x + \log(\sin x) = t$; $1 + \frac{1}{\sin x} \times \cos x\, dx = dt = (1+\cot x)\,dx$

$$\therefore I = \int \frac{dt}{t} = \log|t| + C = \log|x + \log(\sin x)| + C.$$

4) Write the value of $\int \frac{1-\tan x}{x+\log \cos x}\,dx$

Sol^n: put $x + \log \cos x = t$ $\quad \left(1 + \frac{1}{\cos x}(-\sin x)\right)dx = dt$

$(1-\tan x)\,dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + C = \log|x + \log \cos x| + C.$

5) Find the value of $\int \frac{1}{\sin^3 x \cos x}\,dx.$

$I = \int \frac{(\cos^2 x + \sin^2 x)}{\sin^3 x \cdot \cos x}\,dx$ $\quad \left[\text{we know } \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right.$

$= \int \frac{\cos x}{\sin^3 x}\,dx + \int \frac{dx}{\sin x \cos x}$

$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin^2 x}\,dx + \int \frac{2}{2 \sin x \cdot \cos x}\,dx$

$= \int \cot x \cdot \operatorname{cosec}^2 x\,dx + \int \frac{2}{\sin 2x}\,dx$

$= -\int \cot x\,(-\operatorname{cosec}^2 x)\,dx + 2\int \operatorname{cosec} 2x\,dx$

$= -\frac{\cot^2 x}{2} + 2\,\frac{\log|\operatorname{cosec} 2x - \cot 2x| + C}{2}$

$= \log|\operatorname{cosec} 2x - \cot 2x| - \frac{1}{2}\cot^2 x + C.$

6) Evaluate : $\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx$

Sol^n: put $a+b\cos x = t$; $b(-\sin x)\,dx = dt$

$\sin x\,dx = -\frac{dt}{b}$ $\qquad \cos x = \frac{(t-a)}{b}$

$I = \int \frac{2\sin x\cdot\cos x}{(a+b\cos x)^2}\,dx = \int 2\frac{(t-a)}{b}\cdot\frac{(-dt)}{b\cdot t^2}$

$= -\frac{2}{b^2}\int \frac{(t-a)}{t^2}\,dt$

$= -\frac{2}{b^2}\int\left(\frac{1}{t}-\frac{a}{t^2}\right)dt \quad = -\frac{2}{b^2}\left[\log|t| + \frac{a}{t}\right] + c$

$= -\frac{2}{b^2}\left[\log|a+b\cos x| + \frac{a}{a+b\cos x}\right] + c.$

7) Evaluate : $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$

Sol:
$x^4-1 = x^2\left(x^2-\frac{1}{x^2}\right)$

$x^4+x^2+1 = x^2\left(x^2+1+\frac{1}{x^2}\right)$

$\therefore\ I = \int \frac{\cancel{x^2}\left(x^2-\frac{1}{x^2}\right)}{\cancel{x^2}\sqrt{x^2\left(x^2+\frac{1}{x^2}+1\right)}}\,dx = \int \frac{x^2-\frac{1}{x^2}}{x\sqrt{\left(x^2+\frac{1}{x^2}\right)+1}}\,dx$

$= \int \frac{\cancel{x}\left(x-\frac{1}{x^3}\right)dx}{\cancel{x}\sqrt{\left(x^2+\frac{1}{x^2}\right)+1}} = \int \frac{\left(x-\frac{1}{x^3}\right)}{\sqrt{\left(x^2+\frac{1}{x^2}\right)+1}}\,dx$

put $x^2+\frac{1}{x^2} = t$; $\left(2x-\frac{2}{x^3}\right)dx = dt \Rightarrow 2\left(x-\frac{1}{x^3}\right)dx = dt$

$\left(x-\frac{1}{x^3}\right)dx = \frac{dt}{2}$

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{t+1}} = \frac{1}{2}\int (t+1)^{-\frac{1}{2}}\, dt$$

$$= \frac{1}{\cancel{2}} \frac{(t+1)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C = \sqrt{t+1} + C = \sqrt{x^2+\frac{1}{x^2}+1} + C.$$

8) Evaluate : $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

Sol^n^: $\sqrt{x} = \sin\theta \qquad x = \sin^2\theta$

$dx = 2\sin\theta \cdot \cos\theta \, d\theta$

$$I = \int \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\; 2\sin\theta \cdot \cos\theta\, d\theta$$

$$= \int \sqrt{\frac{(1-\sin\theta)^2}{1^2-\sin^2\theta}}\; \sin 2\theta\, d\theta = \int \frac{1-\sin\theta}{\cancel{\cos\theta}} \cdot 2\sin\theta \cdot \cancel{\cos\theta}\, d\theta$$

$$= \int 2(1-\sin\theta)\sin\theta\, d\theta = 2\int (\sin\theta - \sin^2\theta)\, d\theta$$

$$= \int 2\sin\theta - \int 2\sin^2\theta \, d\theta$$

$$= \int 2\sin\theta - \int (1-\cos 2\theta)\, d\theta$$

$$= 2\int \sin\theta\, d\theta - \int d\theta + \int \cos 2\theta\, d\theta$$

$$= -2\cos\theta - \theta + \frac{\sin 2\theta}{2} + c$$

$$= \frac{\cancel{2}\sin\theta \cdot \cos\theta}{\cancel{2}} - 2\cos\theta - \theta + c$$

$$= \sin\theta \cdot \cos\theta - 2\cos\theta - \theta + c.$$

9) Evaluate : $\int \frac{\cos(x+a)}{\sin(x+b)}\,dx$

$$x+a = (x+b) + (a-b)$$

$$\cos(x+a) = \cos\{(x+b)+(a-b)\}$$
$$= \cos(x+b)\cos(a-b) - \sin(x+b)\cdot\sin(a-b)$$

$$\frac{\cos(x+a)}{\sin(x+b)} = \frac{\cos(x+b)\cos(a-b)}{\sin(x+b)} - \frac{\cancel{\sin(x+b)}\sin(a-b)}{\cancel{\sin(x+b)}}$$
$$= \cos(a-b)\cot(x+b) - \sin(a-b)$$

$$\therefore I = \cos(a-b)\int \cot(x+b) - \int \sin(a-b)\,dx$$

$$= \cos(a-b)\log|\sin(x+b)| - x\sin(a-b) + C.$$

10) Evaluate : $\int \frac{dx}{\cos(x+a)\sin(x+b)}$

Solⁿ: $a-b = (x+a) - (x+b)$

$$\cos(a-b) = \cos\{(x+a) - (x+b)\}$$
$$= \cos(x+a)\cos(x+b) + \sin(x+a)\cdot\sin(x+b)$$

$$\therefore I = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\cos(x+a)\sin(x+b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int\left(\frac{\cancel{\cos(x+a)}\cos(x+b)}{\cancel{\cos(x+a)}\sin(x+b)} + \frac{\sin(x+a)\cdot\cancel{\sin(x+b)}}{\cos(x+a)\cancel{\sin(x+b)}}\right)dx$$

$$= \frac{1}{\cos(a-b)}\int\{\cot(x+b) + \tan(x+a)\}\,dx$$

$$= \frac{1}{\cos(a-b)}\{\log|\sin(x+b)| + \log|\sec(x+a)|\} + C.$$

11) Evaluate : $\int \sqrt{\frac{1-x}{1+x}}\,dx$

Sol^n: $\int \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\,dx = \int \frac{(1-x)}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}}$

$\left[\because \int \frac{dx}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c\right]$ $\qquad I = I_1 - I_2$

$I_1 = \sin^{-1}x \qquad I_2 = \int \frac{x\,dx}{\sqrt{1-x^2}}$ $\qquad$ put $1-x^2 = t$

$-2x\,dx = dt$

$x\,dx = \frac{-dt}{2}$

$I_2 = \frac{1}{2}\int \frac{-dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\,dt = -\frac{1}{2}\frac{t^{1/2}}{1/2}$

$= -\sqrt{t} = -\sqrt{1-x^2}$

$\therefore \quad I = \sin^{-1}x + \sqrt{1-x^2} + C.$

12) Evaluate : $\int \frac{dx}{\sqrt{\{\sin^3 x \cos(x-\alpha)\}}}$

Sol^n: $\sin^3 x \cos(x-\alpha) = \sin^3 x\,\{\cos x\cdot\cos\alpha + \sin x\cdot\sin\alpha\}$

$= \sin^4 x\left\{\frac{\cos x}{\sin x}\cdot\cos\alpha + \frac{\sin x}{\sin x}\cdot\sin\alpha\right\}$

$= \sin^4 x\,(\cot x\cdot\cos\alpha + \sin\alpha)$

$\therefore \quad I = \int \frac{dx}{\sin^2 x\sqrt{\cot x\cdot\cos\alpha + \sin\alpha}} = \int \frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\cos\alpha}\sqrt{\cot x + \tan\alpha}}$

put $\sqrt{\cot x + \tan\alpha} = t \Rightarrow \cot x + \tan\alpha = t^2 \Rightarrow -\operatorname{cosec}^2 x\,dx = 2t\,dt$

$I = \frac{1}{\sqrt{\cos\alpha}}\int \frac{-2\cancel{t}\,dt}{\cancel{t}} = -\frac{2}{\sqrt{\cos\alpha}}\int dt = \frac{1}{\sqrt{\cos\alpha}}\,t + C$

$= -\frac{2}{\sqrt{\cos\alpha}}\sqrt{\cot x + \tan\alpha} + C.$

Integrals of the form $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$

i) To integrate $\int \sin^n x\, dx$, $\int \cos^n x\, dx$

To evaluate $\int \sin^n x\, dx$, detach $\sin x$ and in resulting even power of $\sin x$, replace $\sin^2 x$ by $(1-\cos^2 x)$ and then put $\cos x = t$.

Similarly, to evaluate $\int \cos^n x\, dx$, detach $\cos x$ and in the resulting even power of $\cos x$, replace $\cos^2 x$ by $(1-\sin^2 x)$ and then put $\sin x = t$.

Here we express $\sin^n x$ and $\cos^n x$ in terms of sines and cosines of multiples of x by the application of following trigonometric identities:

(i) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ (ii) $\cos^2 x = \frac{1}{2}(1+\cos x)$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ (iv) $4\cos^3 x - 3\cos x$.

ii) $\int \sin^m x \cdot \cos^n x\, dx$, where either m or n is a positive odd integer, whatever the other may be.

Rule:

(i) If m is odd, detach $\sin x$ and resulting even power of $\sin x$, replace $\sin^2 x$ by $(1-\cos^2 x)$ and put $\cos x = t$

(ii) In n is odd, detach $\cos x$ and the resulting even power of $\cos x$ replace $\cos^2 x$ by $(1-\sin^2 x)$ and put $\sin x = t$.

iii) To integrate $\int \sin mx \cos nx\, dx$, $\int \cos mx \cdot \sin nx\, dx$, $\int \cos mx \cdot \cos nx$, $\int \sin mx \cdot \sin nx\, dx$

Use: (i) $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

Examples

1) Find the following integrals:

(i) $\int \sin^5 x \, dx$ (ii) $\int \cos^5 x \, dx$ (iii) $\int \sin^5 x \cdot \cos^4 x \, dx$

Sol^n: (i) $I = \int \sin^5 x \, dx = \int \sin^4 x \cdot \sin x \cdot dx$

$I = \int (\sin^2 x)^2 \sin x \, dx = \int (1-\cos^2 x)^2 \sin x \, dx$

put $\cos x = t$ $\quad -\sin x \, dx = dt \Rightarrow \sin x \, dx = -dt$

$I = \int (1-t^2)^2 (-dt) = -\int (t^4+1-2t^2) dt$

$= -\left[\frac{t^5}{5} + t - 2\frac{t^3}{3} \right] + C = -\frac{\cos^5 x}{5} - \cos x + \frac{2}{3}\cos^3 x + C.$

(ii) $\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx = \int (1-\sin^2 x)^2 \cos x \, dx$

put $\sin x = t$ $\quad \cos x \, dx = dt$

$I = \int (1-t^2)^2 dt = \int (t^4+1-2t^2) dt = \frac{t^5}{5} + t - \frac{2}{3}t^3 \Big] + C$

$= \frac{\sin^5 x}{5} + \sin x - \frac{2}{3}\sin^3 x + C.$

(iii) $I = \int \sin^5 x \cdot \cos^4 x \, dx = \int \sin^4 x \cdot \cos^4 x \sin x \, dx$

$= \int \cos^4 x (1-\cos^2 x)^2 \sin x \, dx$ put $\cos x = t$ $\quad \sin x \, dx = -dt$

$I = -\int t^4 (1-t^2)^2 dt = -\int t^4 (t^4 - 2t^2 + 1) dt$

$= -\int (t^8 - 2t^6 + t^4) dt = -\left(\frac{t^9}{9} - \frac{2}{7}t^7 + \frac{t^5}{5} \right) + C$

$= -\frac{\cos^9 x}{9} + \frac{2}{7}\cos^7 x - \frac{\cos^5 x}{5} + C.$

2) Evaluate $\int \sin^2 x \, dx$

Sol^n: We know $2\sin^2 x = 1 - \cos 2x$

$$\therefore I = \int \frac{(1-\cos 2x)}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{x}{2} - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

3) Obtain $\int \cos mx \cdot \cos nx \, dx$, where m and n are distinct positive integers. What will happen if $m = n$?

Sol^n: We know $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$$\therefore I = \frac{1}{2}\int \{\cos(m+n)x + \cos(m-n)x\} dx$$

$$= \frac{1}{2}\left\{\frac{\sin(m+n)}{m+n} + \frac{\sin(m-n)}{m-n}\right\} + C \quad \text{where } m \neq n$$

When $m = n$ $\quad I = \int \cos^2 mx \, dx = \frac{1}{2}\int (1+\cos 2mx) dx$

$$\therefore I = \int \frac{dx}{2} + \frac{1}{2}\int \cos 2mx \, dx \qquad [\because 2\cos^2 A = 1 + \cos 2A]$$

$$= \frac{x}{2} + \frac{1}{2}\frac{\sin 2mx}{2m} + C = \frac{x}{2} + \frac{\sin 2mx}{4m} + C$$

4) Evaluate : $\int \tan x \cdot \tan 2x \cdot \tan 3x \, dx$

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\tan 3x (1 - \tan 2x \cdot \tan x) = \tan 2x + \tan x$$

$$\therefore \tan 3x \cdot \tan 2x \cdot \tan x = \tan 3x - \tan 2x - \tan x$$

$\therefore I = \int (\tan 3x - \tan 2x - \tan x)\,dx$

$= \frac{1}{3}\log|\cos 3x| - \frac{1}{2}\log|\cos 2x) + \log|\cos x| + C.$

x) Evaluate : $\int \cos^3 x \cdot e^{\log \sin^3 x}\,dx$

$I = \int \cos^3 x \,\sin^3 x\,dx \qquad \left[\because e^{\log x} = x\right]$

$= \frac{1}{8}\int (2\cos x \cdot \sin x)^3\,dx = \frac{1}{8}\int \sin^3 2x\,dx$

$\sin 3A = 3\sin A - 4\sin^3 A \quad \therefore \sin^3 A = -\frac{1}{4}(3\sin A - \sin 3A)$

$\therefore I = \frac{1}{8}\int -\frac{1}{4}(3\sin 2x - \sin 6x)\,dx$

$= \frac{3}{32}\int \sin 2x - \frac{1}{32}\int \sin 6x\,dx$

$= \frac{3}{32}\left(-\frac{\cos 2x}{2}\right) - \frac{1}{32}\left(-\frac{\cos 6x}{6}\right) + C$

$= -\frac{3}{64}\cos 2x + \frac{1}{192}\cos 6x + C.$

s) Evaluate : $\int \cos^4 x\,dx$

Sol^n $\quad I = \int (\cos^2 x)^2\,dx = \int \left\{\frac{1+\cos 2x}{2}\right\}^2 dx$

$I = \frac{1}{4}\int (1 + \cos^2 2x + 2\cos 2x)\,dx$

$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$

$= \frac{3}{8}\int dx + \frac{1}{2}\int \cos 2x + \frac{1}{8}\int \cos 4x\,dx$

$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$

Some Special integrals

Two Special integrals: 1.) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c \quad x>a.$

2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c \quad x<a.$

Useful Substitutions

Expression:	Substitution:
(i) a^2-x^2	$x = a\sin\theta$ (or $x = a\cos\theta$)
(ii) a^2+x^2	$x = a\tan\theta$ (or $x = a\cot\theta$)
(iii) x^2-a^2	$x = a\sec\theta$ (or $x = a\,\mathrm{cosec}\,\theta$)
(iv) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(v) $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $(x-\alpha)(\beta-x)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

4) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

5) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

6) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$

7) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$

8) $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

9) $\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

10) $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

Solved Examples

1) Write the value of $\int \frac{dx}{x^2+16}$.

Sol^n: $I = \int \frac{dx}{x^2+16} = \int \frac{dx}{x^2+(4)^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$.

2) Integrate the following w.r.t. x : $\frac{x^2-3x+1}{\sqrt{1-x^2}}$

Sol^n: $I = \int \frac{x^2-3x+1}{\sqrt{1-x^2}}\,dx$

$= \int \frac{2-(1-x^2)-3x}{\sqrt{1-x^2}}\,dx = \int \frac{2}{\sqrt{1-x^2}}\,dx - \int\sqrt{1-x^2}\,dx - 3\int\frac{x\,dx}{\sqrt{1-x^2}}\,dx$

$= 2\cdot \sin^{-1}\frac{x}{1} - \left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right) - I_1$

$I_1 = 3\int\frac{x}{\sqrt{1-x^2}}\,dx$ put $1-x^2 = t$ $\quad -2x\,dx = dt$

$x\,dx = -\frac{dt}{2}$

$= -\frac{3}{2}\int\frac{dt}{\sqrt{t}} = -\frac{3}{2}\int (t)^{-1/2}\,dt$

$= -\frac{3}{2}\frac{(t)^{1/2}}{1/2} = -3\sqrt{1-x^2}$

$\therefore I = 2\sin^{-1}x - \frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x + 3\sqrt{1-x^2} + C$

3) Integrate $\int\frac{x^3}{\sqrt{1-x^8}}\,dx$

Sol^n: put $x^4 = t$ $\quad 4x^3\,dx = dt$

$I = \frac{1}{4}\int\frac{dt}{\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1}t + C = \frac{1}{4}\sin^{-1}(x^4) + C.$

4) Find $= \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\, dx$

Soln: $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\, dx$

put $x^{3/2} = t$ $\quad \frac{3}{2} x^{1/2}\, dx = dt \Rightarrow \sqrt{x}\, dx = \frac{2}{3}\, dt$

$x^3 = t^2$

$I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1} \frac{x^{3/2}}{a^{3/2}} + C \quad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c\right]$

$= \frac{2}{3} \sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}} + C.$

5) Evaluate : $\int \frac{\sqrt{16+(\log x)^2}}{x}\, dx$

Soln: $I = \int \frac{\sqrt{16+(\log x)^2}}{x}\, dx$

Put $\log x = t \Rightarrow \frac{1}{x}\, dx = dt$

$I = \int \sqrt{16+t^2}\, dt \Rightarrow \int \sqrt{(4)^2+t^2}\, dt$

$= \frac{t}{2}\sqrt{16+t^2} + \frac{16}{2} \log\left| t + \sqrt{16+t^2}\right| + C$

$= \frac{\log x}{2}\sqrt{16+(\log x)^2} + 8 \log\left|(\log x) + \sqrt{16+(\log x)^2}\right| + C.$

6) Find : $\int \frac{\sin 2x \cos 2x}{\sqrt{9-\cos^4(2x)}}\,dx$

Sol^n: $I = \int \frac{\sin 2x \cos 2x}{\sqrt{9-\cos^4(2x)}}\,dx$

$\cos^2(2x) = t \Rightarrow 2\cos 2x(-\sin 2x)2\,dx = dt$

$\therefore \quad -4\sin 2x \cos 2x\,dx = dt$

$\therefore \quad \sin 2x \cos 2x\,dx = -\frac{1}{4}dt$

$I = -\frac{1}{4}\int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4}\int \frac{dt}{\sqrt{3^2-t^2}} = -\frac{1}{4}\sin^{-1}\frac{t}{3} + c$

$= -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + c$

7) Evaluate : $\int \frac{\cos 2x}{1+4\sin^2 x \cos^2 x}\,dx$

Sol^n :

$I = \int \frac{\cos 2x}{1+(2\sin x \cos x)^2}\,dx \qquad [\because 2\sin x \cos x = \sin 2x]$

$= \int \frac{\cos 2x}{1+\sin^2 2x}\,dx \qquad$ put $\sin 2x = t$; $2\cos 2x\,dx = dt$

$= \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\cdot\tan^{-1} t + c = \frac{1}{2}\tan^{-1}(\sin 2x) + c.$

8) Evaluate: $\int \sqrt{\frac{x+a}{x-a}}\,dx$

Sol^n: $I = \int \sqrt{\frac{x+a}{x-a}}\,dx = \int \sqrt{\frac{(x+a)(x+a)}{(x-a)(x+a)}}\,dx$

$= \int \frac{x+a}{\sqrt{x^2-a^2}}\,dx = \int \frac{x}{\sqrt{x^2-a^2}}\,dx + \int \frac{a}{\sqrt{x^2-a^2}}\,dx$

$= I_1 + aI_2$

$I_1 = \int \frac{x\,dx}{\sqrt{x^2-a^2}}$ Put $\sqrt{x^2-a^2} = t \Rightarrow x^2-a^2 = t^2$

$\cancel{2}x\,dx = \cancel{2}t\,dt \Rightarrow x\,dx = t\,dt$

$= \int \frac{\cancel{t}\,dt}{\cancel{t}} = \int dt = t + C$

$I_1 = \sqrt{x^2-a^2} + C_1$

$I_2 = \int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C_2$

$\therefore I = \sqrt{x^2-a^2} + \log|x+\sqrt{x^2-a^2}| + C.$

9) $I = \int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

Sol^n: put $x^2 = t$ $\therefore$ $2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$\therefore I = \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t}}\,dt = \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t} \times \frac{a^2-t}{a^2-t}}\,dt$

$= \frac{1}{2}\int \frac{(a^2-t)}{\sqrt{a^4-t^2}}\,dt = \frac{1}{2}\int \frac{a^2}{\sqrt{a^4-t^2}}\,dt - \frac{1}{2}\int \frac{t\,dt}{\sqrt{a^4-t^2}}$

$= \frac{1}{2}a^2\int \frac{dt}{\sqrt{a^4-t^2}} - \frac{1}{2}\int \frac{t\,dt}{\sqrt{a^4-t^2}}$

$$I = \frac{a^2}{2}\sin^{-1}\frac{t}{a^2} - \frac{1}{2}\int\frac{t\,dt}{\sqrt{a^4+}} = \frac{a^2}{2}\sin^{-1}\frac{x^2}{a^2} - I_2$$

$$I_2 = -\frac{1}{2}\int\frac{t\,dt}{\sqrt{a^4-t^2}}$$

put $a^4 - t^2 = u$

$-2t\,dt = du$

$t\,dt = -\frac{du}{2}$

$$I_2 = \frac{1}{4}\int\frac{du}{\sqrt{u}} = \frac{1}{4}\,2u^{\frac{1}{2}} = \frac{1}{2}\sqrt{a^4-t^2}$$

$$= \frac{1}{2}\sqrt{a^4-x^4}$$

$$\therefore\ I = \frac{a^2}{2}\sin^{-1}\frac{x^2}{a^2} + \frac{1}{2}\sqrt{a^4-x^4} + c.$$

10) Evaluate : $\int\frac{dx}{x\sqrt{9+x^4}}$

Solⁿ: $I = \int\frac{dx}{x\sqrt{9+x^4}} = \int\frac{x^3\,dx}{x^4\sqrt{9+x^4}}$

Put $\sqrt{9+x^4} = t$, $(9+x^4)^{\frac{1}{2}} = t \Rightarrow \frac{1}{2}(9+x^4)^{-1/2}\,4x^3\,dx = dt$

$$\therefore\ dt = \frac{2x^3}{\sqrt{9+x^4}}\,dx \Rightarrow \frac{x^3\,dx}{\sqrt{9+x^4}} = \frac{1}{2}\,dt$$

$$9+x^4 = t^2 \Rightarrow x^4 = t^2-9$$

$$\therefore\ I = \frac{1}{2}\int\frac{dt}{t^2-9} = \frac{1}{2}\int\frac{dt}{t^2-(3)^2} = \frac{1}{2}\,\frac{1}{2\cdot 3}\log\left|\frac{t-3}{t+3}\right| + c$$

$$= \frac{1}{12}\log\left|\frac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right| + c.$$

11) Evaluate : $\int \frac{ax^3+bx}{x^4+c^2}\,dx$

Sol^n : $I = \int \frac{ax^3+bx}{x^4+c^2}\,dx = a\int \frac{x^3\,dx}{x^4+c^2} + b\int \frac{x\,dx}{x^4+c^2}$

$I = aI_1 + bI_2$

$\therefore I_1 = \int \frac{x^3\,dx}{x^4+c^2}$ Put $x^4+c^2 = t$ $4x^3\,dx = dt$

$\therefore x^3\,dx = \frac{1}{4}\,dt$

$I_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C_1 = \frac{1}{4}\log|x^4+c^2| + C_1$

$I_2 = \int \frac{x\,dx}{x^4+c^2}$ put $x^2 = u$ $2x\,dx = du \Rightarrow x\,dx = \frac{du}{2}$

$\therefore I_2 = \frac{1}{2}\int \frac{du}{u^2+c^2} = \frac{1}{2}\,\frac{1}{c}\tan^{-1}\frac{u}{c} + C_2 = \frac{1}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + C_2$

$\therefore I = \frac{a}{4}\log|x^4+c^2| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + k$ $(k = C_1 + C_2)$.

12) Evaluate : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}\,dx$

Sol^n : $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}\,dx$ put $\tan x = t$

$\sec^2 x\,dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t^2+(2)^2}} = \log|t+\sqrt{t^2+4}| + C$

$I = \log|\tan x + \sqrt{\tan^2 x+4}| + C.$

13) Evaluate : $\int \frac{dx}{1+3\sin^2 x}$

Sol^n : $\quad I = \int \frac{dx}{1+3\sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$

$\therefore \quad I = \int \frac{\sec^2 x \, dx}{\sec^2 x + 3\tan^2 x} \qquad [\because 1+\tan^2 x = \sec^2 x]$

$= \int \frac{\sec^2 x \, dx}{1+\tan^2 x + 3\tan^2 x} = \int \frac{\sec^2 x \, dx}{1+4\tan^2 x}$

Put $\tan x = t \qquad \sec^2 x \, dx = dt$

$\therefore \quad I = \int \frac{dt}{1+4t^2} = \frac{1}{4}\int \frac{dt}{\left(\frac{1}{2}\right)^2 + t^2} = \frac{1}{4} \frac{1}{1/2} \tan^{-1} \frac{t}{1/2} + c$

$= \frac{1}{2} \tan^{-1}(2t) + c = \frac{1}{2} \tan^{-1}(2\tan x) + c.$

14) Evaluate : $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

Sol^n : $\quad I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

Dividing numerator and denominator by $\cos^2 x$

$\therefore \quad I = \int \frac{\sec^2 x \, dx}{a^2 \tan^2 x + b^2} \qquad$ put $\tan x = t$

$\sec^2 x \, dx = dt$

$\therefore \quad I = \int \frac{dt}{b^2 + a^2 t^2} = \frac{1}{a^2} \int \frac{dt}{\left(\frac{b}{a}\right)^2 + t^2}$

$= \frac{1}{a^2} \cdot \frac{a}{b} \tan^{-1} \frac{at}{b} + c = \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan x\right) + c.$

15) Evaluate: $\int \frac{\sin x}{\sin 3x}\,dx$ Ans. $\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x} \right| + C$

[HINT. $\sin 3x = 3\sin x - 4\sin^3 x$

16) Evaluate: $\int \frac{\cos x}{\cos 3x}\,dx$ Ans. $\frac{1}{2\sqrt{3}} \log \left| \frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x} \right| + C$

17) Evaluate: $\int \frac{dx}{9\cos^2 x + 16\sin^2 x}$ Ans $\frac{1}{12} \tan^{-1}\left(\frac{4\tan x}{3}\right) + C$

18) Evaluate: $\int \frac{dx}{\sin^2 x + \tan^2 x}$ Ans. $\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$

19) Evaluate: $\int \frac{dx}{\sin^2 x + 2\cos^2 x + 3}$ Ans. $\frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C.$

20) Evaluate: $\int \frac{dx}{x\sqrt{\{(\log x)^2 + 10\}}}$ Ans. $\log\left|\log x + \sqrt{\{(\log x)^2 + 10\}}\right| + C$

Guidlines

Integral of the form: $\int \frac{1}{a\sin^2 x + b\cos^2 x}\,dx$, $\int \frac{1}{a+b\cos^2 x}\,dx$

$\int \frac{1}{a+b\sin^2 x}\,dx$, $\int \frac{1}{(a\sin x + b\cos x)^2}\,dx$, $\int \frac{1}{a+b\sin^2 x + c\cos^2 x}\,dx$

(i) Divide the numerator and denominator by $\cos^2 x$.

(ii) Replace $\sec^2 x$, if any, in the denominator by $1+\tan^2 x$.

(iii) Put $\tan x = t$ so that $\sec^2 x\,dx = dt$.

Integration by Parts

When the integrand is the product of two functions, the integral can, in some case, be expressed in the form of a simple integral, which can be evaluated by the Rule of Parts, stated below:

Rule:

Integral of the product of two functions
= First function × Integral of second function
— Integral [(diff. coeff. of the first function) × (Integral of second function)]

$$\Rightarrow \int f_1(x) \cdot f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left[\frac{d f_1(x)}{dx} \cdot \int f_2(x)\, dx \right] dx .$$

Useful Guidances

1) When both the functions in the product can be easily integrated separately and one of them is of the form x^n (n being a positive integer), take this as the first function

2) If one of the two function in the product is a logarithmic or an inverse trigonometric function, it must be taken as the first function.

3) The rule of parts can also be applied to single function and is especially useful in the case of logarithmic and inverse-trigonometric function.
For this, unity (i.e. 1) is taken as the second function.

ILATE METHOD:

I - Inverse Trigometric, L - logarithmic
A - Algebraic function, T - Trigonometric function, E - Exponential.

We may chose the first function, which comes first in the word ILATE.

Examples

1) Evaluate : $\int x e^{2x}\, dx$.

Soln : $\int x e^{2x} dx = x \int e^{2x} dx - \int \left(\frac{dx}{dx} \cdot \int e^{2x} \right) dx$

$$I = \frac{x \cdot e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2}\, dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{2} \frac{e^{2x}}{2} + c$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$$

2) Evaluate $\int \cos\sqrt{x}\, dx$

Soln : put $\sqrt{x} = t$, $x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad I = \int \cos t \; 2t\, dt = 2 \int t \cdot \cos t\, dt$$

$$= 2 \left[t \int \cos t\, dt - \int \left(\frac{dt}{dt} \cdot \int \cos t\, dt \right) dt \right]$$

$$= 2 \left[t (\sin t) - \int 1 \cdot \sin t\, dt \right]$$

$$= 2 \left[t \sin t - (-\cos t) \right] + c$$

$$= 2t \sin t + \cos t + c$$

$$= 2 (\sqrt{x} \sin\sqrt{x} + \cos\sqrt{x}) + c.$$

3) Evaluate $\int x \log(1+x)\,dx$

$$I = \int \log(1+x)\cdot x\,dx = \log(1+x)\int x\,dx - \int\left[\frac{d(1+x)}{dx}\cdot\int x\,dx\right]dx$$

$$= \log(1+x)\frac{x^2}{2} - \int\frac{1}{1+x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}x^2\log(1+x) - \frac{1}{2}\int\frac{x^2}{1+x}\,dx$$

$$I = \frac{1}{2}x^2\log(1+x) - \frac{1}{2}I_1$$

$$I_1 = \int\frac{x^2}{1+x}\,dx = \int\frac{x^2-x+x-1+1}{1+x}\,dx$$

$$= \int\frac{(x^2+x)-(x+1)+1}{1+x}\,dx$$

$$= \int\frac{x(x+1)-(1+x)+1}{1+x}\,dx$$

$$= \int x\,dx - \int dx + \int\frac{dx}{1+x}$$

$$= \frac{x^2}{2} - x + \log|1+x| + C$$

$$\therefore I = \frac{1}{2}x^2\log(1+x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log|1+x| + C.$$

4) Evaluate $\int x\log x^2\,dx$

Soln :
$$I = \int \log x^2\cdot x\,dx = \log x^2\int x\,dx - \int\left(\frac{d(\log x^2)}{dx}\int x\,dx\right)dx$$

$$= \log x^2\,\frac{x^2}{2} - \int\frac{\cancel{2}x}{\cancel{x^2}}\,\frac{\cancel{x^2}}{\cancel{2}}\,dx$$

$$= \frac{1}{2}x^2\log x^2 - \frac{x^2}{2} + C$$

$[\log x^2 = 2\log x]$.

$$= \frac{1}{\cancel{2}}x^2\cdot\cancel{2}\log x - \frac{x^2}{2} + C$$

$$= x^2\log x - \frac{x^2}{2} + C$$

5) Evaluate $\int x^2 \tan^{-1}x \, dx$

Soln : $I = \int \tan^{-1}x \; x^2 \, dx$

$= \tan^{-1}x \int x^2 dx - \int \left(\frac{d(\tan^{-1}x)}{dx} \int x^2 dx \right) dx$

$= \tan^{-1}x \; \frac{x^3}{3} - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} \, dx$

$= \frac{1}{3} x^3 \tan^{-1}x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$

$= \frac{x^3}{3} \tan^{-1}x - \frac{1}{3} I_1$

$I_1 = \int \frac{x^3}{1+x^2} \, dx = \int \frac{x^3+x-x}{1+x^2} \, dx = \int \frac{x(x^2+1)-x}{1+x^2} \, dx$

$= \int \left(x - \frac{x}{1+x^2} \right) dx = \int x \, dx - \int \frac{x}{1+x^2} \, dx$

$= \frac{x^2}{2} - I_2$; $I_2 = \int \frac{x}{1+x^2} \, dx$ put $1+x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2} dt$

$\therefore I_2 = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log|t| + C = \frac{1}{2} \log|1+x^2| + C$

$\therefore I_1 = \frac{x^2}{2} - \frac{1}{2} \log|1+x^2| + C$

$\therefore I = \frac{x^3}{3} \tan^{-1}x - \frac{1}{3} \left(\frac{x^2}{2} - \frac{1}{2} \log|1+x^2| \right) + C$

6) Evaluate : $\int \log x \, dx$

Soln : $I = \int \log x \cdot 1 \, dx = \log x \int dx - \int \left(\frac{d(\log x)}{dx} \int dx \right) dx$

$= x \log x - \int \frac{1}{x} \cdot x \cdot dx$

$= x \log x - x + C.$

7) Evaluate : $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx$.

Sol^n : put $\sin^{-1} x = t$ $\quad x = \sin t$

$dx = \cos t \cdot dt$

$\therefore$

$$I = \int \frac{\sin t \cdot t \cdot \cos t \cdot dt}{\sqrt{1-\sin^2 t}} = \int \frac{t \cdot \sin t \cdot \cancel{\cos t}}{\cancel{\cos t}}\, dt$$

$$= \int t \cdot \sin t \cdot dt = t \int \sin t\, dt - \int \left(\frac{dt}{dt} \cdot \int \sin t\, dt \right) dt$$

$$= -t \cos t - \int \cdot 1 \, (-\cos t)\, dt$$

$$= -t \cos t + \int \cos t\, dt = -t \cos t + \sin t + C$$

$$= x - \sqrt{1-x^2}\, \sin^{-1} x + C$$

8) Evaluate : $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

Sol^n: put $x = a \tan^2 \theta$; $dx = a\, 2 \tan\theta \cdot \sec^2\theta\, d\theta$

$$\therefore I = \int \sin^{-1} \left(\sqrt{\frac{a \tan^2\theta}{a + a\tan^2\theta}} \right) 2a \cdot \tan\theta \cdot \sec^2\theta \cdot d\theta$$

$$= 2a \int \sin^{-1} \left(\frac{\tan\theta}{\sec\theta} \right) \tan\theta \cdot \sec^2\theta\, d\theta$$

$$= 2a \int \sin^{-1} (\sin\theta) \, (\tan\theta \cdot \sec^2\theta)\, d\theta$$

$$= 2a \int \theta \cdot (\tan\theta \cdot \sec^2\theta)\, d\theta$$

$$= 2a \left[\theta \int \tan\theta \cdot \sec^2\theta\, d\theta - \int \left(\frac{d\theta}{d\theta} \cdot \int \tan\theta \cdot \sec^2\theta\, d\theta \right) d\theta \right.$$

$$I_1 = \int \tan\theta \cdot \sec^2\theta\, d\theta$$

$$I_1 = \int \tan\theta \cdot \sec^2\theta \, d\theta$$

put $\tan\theta = y$ $\quad \sec^2\theta \, d\theta = dy$

$$\therefore I_1 = \int y \, dy = \frac{y^2}{2} = \frac{1}{2}\tan^2\theta$$

$$\therefore I = 2a\left[\theta \cdot \frac{1}{2}\tan^2\theta - \int 1 \cdot \frac{\tan^2\theta}{2} d\theta\right]$$

$$= 2a\left[\frac{1}{2}\theta \cdot \tan^2\theta - \frac{1}{2}\int (\sec^2\theta - 1) d\theta\right]$$

$$= 2a\left[\frac{1}{2}\theta\tan^2\theta - \frac{1}{2}(\tan\theta - \theta)\right] + c$$

$$= a\theta\tan^2\theta - a\tan\theta + a\theta + c.$$

$\left[\because x = a\tan^2\theta \quad \tan^2\theta = \frac{x}{a}, \quad \tan\theta = \sqrt{\frac{x}{a}} \quad \theta = \tan^{-1}\sqrt{\frac{x}{a}}\right]$

$$\therefore I = a\tan^{-1}\sqrt{\frac{x}{a}} \cdot \frac{x}{a} - a\sqrt{\frac{x}{a}} + a \cdot \tan^{-1}\sqrt{\frac{x}{a}} + c.$$

$$I = x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c.$$

9) Find : $\displaystyle\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

Sol^n : $\displaystyle I = \int \log(\log x) + \int \frac{dx}{(\log x)^2}$

$$= I_1 + I_2$$

$$I_1 = \int \log(\log x)\, dx \qquad I_2 = \int \frac{1}{(\log x)^2} dx$$

$$I_1 = \int \log(\log x)\cdot 1\,dx = \log(\log x)\int dx - \int\left(\frac{d}{dx}(\log(\log x))\cdot\int dx\right)dx$$

$$= \log(\log x)\,x - \int \frac{1}{\log x}\cdot\frac{1}{\cancel{x}}\cdot\cancel{x}\,dx$$

$$= x\log(\log x) - \int \frac{1}{\log x}\,dx.$$

$$= x\log(\log x) - I_3$$

$$I_3 = \int \frac{1}{\log x}\cdot 1\,dx = \frac{1}{\log x}\cdot x - \int \frac{d}{dx}\left(\frac{1}{\log x}\right)\cdot x\cdot dx.$$

$$\frac{d}{dx}\left(\frac{1}{\log x}\right) = \frac{-1\cdot\frac{d(\log x)}{dx} + \log x\,\frac{d(1)}{dx}}{(\log x)^2} = \frac{\frac{1}{x} - 0}{(\log x)^2} = -\frac{1}{x(\log x)^2}$$

$$I_3 = \frac{x}{\log x} - \int \frac{1}{x(\log x)^2}\cdot x\cdot dx = \frac{x}{\log x} + \int \frac{1}{(\log x)^2}\,dx$$

$$\therefore I = x\log(\log x) - \frac{x}{\log x} - \cancel{\int \frac{1}{(\log x)^2}\,dx} + \cancel{\int \frac{dx}{(\log x)^2}\,dx}$$

$$I = x\log(\log x) - \frac{x}{\log x} + C.$$

$$\left[\because \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d(f(x))}{dx} - f(x)\frac{d(g(x))}{dx}}{[g(x)]^2}\right]$$

10) Evaluate : (i) $\int \log (1+x)^{1+x} \, dx$ (ii) $\int \log (2+x^2) \, dx$

Soln : $I = \int \log (1+x)^{1+x} \cdot dx$

$$I = \int (1+x) \log (1+x) \, dx = \int \log (1+x) \, (1+x) \, dx$$

$$= \log (1+x) \int (1+x) dx - \int \left[\frac{d(1+x)}{dx} \cdot \int (1+x) dx \right] dx$$

$$= \log (1+x) \left(x + \frac{x^2}{2} \right) - \int 1 (1+x) \, dx$$

$$= \log (1+x) \left(x + \frac{x^2}{2} \right) - \left(x + \frac{x^2}{2} \right) + C$$

$$= \frac{x(x+1)}{2} \left[\log (1+x) - 1 \right] + C$$

(ii) $I = \int \log (2+x^2) \, dx$

$$I = \int \log (2+x^2) \cdot 1 \, dx = \log (2+x^2) \int dx - \int \left[\frac{d(\log (2+x^2))}{dx} \int dx \right] dx$$

$$= \log(2+x^2) \, x - \int \frac{1}{2+x^2} \cdot 2x \cdot x \cdot dx$$

$$= x \log (2+x^2) - \int \frac{2x^2}{2+x^2} \, dx$$

$$= x \log (2+x^2) - 2 \int \frac{x^2+2-2}{x^2+2} \, dx$$

$$= x \log (2+x^2) - 2 \int \left(1 - \frac{2}{x^2+2} \right) dx$$

$$= x \log (2+x^2) - 2x + 2\sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + C .$$

(11) Evaluate : $\int x \sin^{-1} x \, dx$

Sol^n: $I = \sin^{-1}x \int x\,dx - \int \left[\frac{d(\sin^{-1}x)}{dx} \cdot \int x \cdot dx \right] dx$

$= \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \frac{x^2}{2} dx$

$= \frac{x^2}{2} \sin^{-1}x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2} \sin^{-1}x + \frac{1}{2} \int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2} \sin^{-1}x - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \sqrt{1-x^2}\, dx$

$I = \frac{x^2}{2} \sin^{-1}x - \frac{1}{2} \sin^{-1}x + \frac{1}{2} I_1$

$I_1 = \int \sqrt{1-x^2}\, dx \qquad x = \sin\theta \Rightarrow dx = \cos\theta \cdot d\theta$

$I_1 = \int \sqrt{1-\sin^2\theta} \cos\theta \cdot d\theta = \frac{1}{2} \int 2\cos\theta \cdot \cos\theta \, d\theta$

$= \frac{1}{2} \int (1+\cos 2\theta) d\theta = \frac{\theta}{2} + \frac{1}{2} \frac{\sin 2\theta}{2} + c$

$= \frac{1}{2} \sin^{-1}x + \frac{1}{4} \, 2\sin\theta \cdot \cos\theta$

$= \frac{1}{2} \sin^{-1}x + \frac{1}{2} x \cdot \sqrt{1-x^2}$

$\therefore I = \frac{x^2}{2} \sin^{-1}x - \frac{1}{2} \sin^{-1}x + \frac{1}{4} \sin^{-1}x + \frac{1}{4} x \cdot \sqrt{1-x^2} + c,$

$= \frac{x^2}{2} \sin^{-1}x + \frac{1}{4} (x\sqrt{1-x^2} - \sin^{-1}x) + c.$

Integration by parts.

Integrals of the form $\int e^{ax}\cdot\sin bx\cdot dx$ and $\int e^{ax}\cdot\cos bx\cdot dx$.

12) Evaluate : (i) $\int e^{ax}\cos bx\, dx$ (ii) $\int e^{ax}\sin bx\cdot dx$ (iii) $\int e^{ax}\cos(bx+c)\,dx$

Solⁿ: $I = \int e^{ax}\cos bx\cdot dx$

$$I = \cos bx\cdot dx \int e^{ax}\cdot dx - \int\left[\frac{d(\cos bx)}{dx}\cdot\int e^{ax}\cdot dx\right]dx$$

$$= \cos bx\cdot\frac{e^{ax}}{a} - \int -\sin bx\cdot b\ \frac{e^{ax}}{a}\cdot dx$$

$$I = \frac{1}{a}e^{ax}\cdot\cos bx + \int\frac{b}{a}\ e^{ax}\cdot\sin bx\cdot dx$$

$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left[\sin bx\cdot\frac{e^{ax}}{a} - \int b\cos bx\cdot\frac{e^{ax}}{a}dx\right]$$

$$I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}\sin bx\ e^{ax} - \frac{b^2}{a^2}\int e^{ax}\cdot\cos bx\cdot dx$$

$$I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}\sin bx\, e^{ax} - \frac{b^2}{a^2}I$$

$$I\left(1+\frac{b^2}{a^2}\right) = \frac{1}{a^2}\left(a e^{ax}\cos bx + b\sin bx\, e^{ax}\right)$$

$$I\left(\frac{a^2+b^2}{\cancel{a^2}}\right) = \frac{e^{ax}}{\cancel{a^2}}\left(a\cos bx + b\sin bx\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + c,$$

(ii) Ans : $I = \dfrac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + C$

(iii) $I = \int e^{ax} \cos(bx+c)\, dx$

Integration by parts

$$I = e^{ax} \int \cos(bx+c)\, dx - \int \left(\frac{d(e^{ax})}{dx} \cdot \int \cos(bx+c)\right) dx$$

$$= e^{ax} \frac{\sin(bx+c)}{b} - \int a \cdot e^{ax} \frac{\sin(bx+c)}{b}\, dx$$

$$I = \frac{e^{ax} \sin(bx+c)}{b} - \frac{a}{b} \int e^{ax} \cdot \sin(bx+c)\, dx$$

$$I = \frac{e^{ax} \sin(bx+c)}{b} + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} \int e^{ax} \cos(bx+c)\, dx$$

$$\left(1 + \frac{a^2}{b^2}\right) I = \frac{e^{ax}}{b^2} [b \sin(bx+c) + a \cos(bx+c)]$$

$$I = \frac{e^{ax}}{a^2+b^2} [b \sin(bx+c) + a \cos(bx+c)] + C_1$$

13) Evaluate $\int \sin(\log x)\, dx$

$$I = \int \sin(\log x)\, dx = \int \sin(\log x) \cdot 1\, dx$$

$$= \sin(\log x) \int dx - \int \left(\frac{d[\sin(\log x)]}{dx} \int dx\right) dx$$

$$= x \sin(\log x) - \int \frac{\cos(\log x)}{\cancel{x}} \cdot \cancel{x} \cdot dx$$

$$= x \sin(\log x) - \int \cos(\log x) \cdot 1\, dx$$

$$= x \sin(\log x) - \left[\cos(\log x) \cdot x - \int \frac{-\sin(\log x)}{\cancel{x}} \cdot \cancel{x} \cdot dx\right]$$

$$= x \sin(\log x) - x \cos(\log x) - \int \sin(\log x)\, dx$$

$$I = x \{\sin(\log x) - \cos(\log x)\} - I$$

$$\therefore I = \frac{x}{2} \{\sin(\log x) - \cos(\log x)\} + C$$

14) Evaluate: $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$

Sol^n: $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}} \cdot dx$

put $x = \cos\theta$

$$\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\therefore \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1}{2}\cos^{-1}x$$

$$\therefore I = \frac{1}{2}\int \cos^{-1}x \, dx = \frac{1}{2}\int \cos^{-1}x \cdot 1\, dx$$

$$= \frac{1}{2}\cos^{-1}x \int dx - \frac{1}{2}\int\left(\frac{d(\cos^{-1}x)}{dx} \cdot \int dx\right) dx$$

$$= \frac{1}{2}\cos^{-1}x \cdot x - \frac{1}{2}\int \frac{-1}{\sqrt{1-x^2}} \cdot x \cdot dx$$

$$= \frac{1}{2} x \cdot \cos^{-1}x + \frac{1}{2}\int \frac{x \cdot dx}{\sqrt{1-x^2}}$$

$$I = \frac{x}{2}\cos^{-1}x + \frac{1}{2} I_1$$

$$I_1 = \int \frac{x \cdot dx}{\sqrt{1-x^2}}$$ put $1-x^2 = t \Rightarrow -2x \cdot dx = dt$

$x \cdot dx = -\frac{dt}{2}$

$$= \frac{-1}{2}\int \frac{dt}{\sqrt{t}} = \frac{-1}{2}\int t^{-1/2}\, dt = \frac{-1}{2} \frac{t^{1/2}}{1/2} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$\therefore I = \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C.$$

(15) Evaluate $\int x \tan^{-1}(2x+3)\,dx$

Ans: $\left(\frac{x^2}{2}-1\right)\tan^{-1}(2x+3)-\frac{x}{4}+\frac{3}{8}\log|2x^2+6x+5|+c.$

16) Evaluate: $\int \left[\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\right]dx$

Soln: We know $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$ $\quad \cos^{-1}\sqrt{x}=\frac{\pi}{2}-\sin^{-1}\sqrt{x}.$

$$\therefore I = \int \frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\,dx$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x} - \int dx$$

$$I = \frac{4}{\pi}\int \sin^{-1}\sqrt{x} - x$$

$\sin^{-1}\sqrt{x}=\theta \Rightarrow \sqrt{x}=\sin\theta \Rightarrow x=\sin^2\theta$

$dx = 2\sin\theta\cdot\cos\theta\,d\theta \Rightarrow dx=\sin 2\theta\cdot d\theta$

$$\int \sin^{-1}\sqrt{x}\,dx = \int \theta\cdot\sin 2\theta\cdot d\theta$$

$$= \theta\int\sin 2\theta\,d\theta - \int\left(\frac{d\theta}{d\theta}\int\sin 2\theta\,d\theta\right)d\theta$$

$$= \frac{\theta(-\cos 2\theta)}{2} - \int 1\cdot\left(-\frac{\cos 2\theta}{2}\right)d\theta$$

$$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{2}\frac{\sin 2\theta}{2} + c$$

$$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta + C$$

$$= -\frac{\theta}{2}(1-2\sin^2\theta) + \frac{1}{2}\sin\theta\cdot\cos\theta + C$$

$$= -\frac{\sin^{-1}\sqrt{x}}{2}(1-2x) + \frac{1}{2}\sqrt{x}\left(\sqrt{1-\sin^2\theta}\right) + C$$

$$= -\frac{\sin^{-1}\sqrt{x}}{2}(1-2x) + \frac{1}{2}\sqrt{x}\,(\sqrt{1-x} + C.$$

$$I = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x$$

$$= \frac{4}{\pi}\left\{-\frac{1}{2}\sin^{-1}\sqrt{x}\,(1-2x) + \frac{1}{2}\sqrt{x}\cdot\sqrt{1-x}\right\} - x + c.$$

$$= \frac{2}{\pi}\left\{\sqrt{1-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right\} - x + c.$$

(17) Evaluate : $\displaystyle\int \cos 2x \cdot \log\frac{\cos x + \sin x}{\cos x - \sin x}\,dx$

Sol^n: Integrating by parts and taking $\log\dfrac{\cos x + \sin x}{\cos x - \sin x}$ as first function and $\cos 2x$ as second function.

$$I = \log\frac{\cos x+\sin x}{\cos x-\sin x}\int \cos 2x\cdot dx - \int\left(\frac{d}{dx}\left[\log\frac{\cos x+\sin x}{\cos x-\sin x}\right]\cdot\int\cos 2x\,dx\right)dx$$

$$= \log\frac{\cos x+\sin x}{\cos x-\sin x}\cdot\frac{\sin 2x}{2} - \int\frac{\cos x-\sin x}{\cos x+\sin x}\,\frac{d}{dx}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)\cdot\frac{\sin 2x}{2}\,dx$$

$$= \sin x\cdot\cos x\,\log\frac{\cos x+\sin x}{\cos x-\sin x} - \int\frac{\cos x-\sin x}{\cos x+\sin x}\left\{\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}\right\}\cdot\frac{\sin 2x}{2}\,dx$$

$$= \sin x\cdot\cos x\cdot\log\frac{\cos x+\sin x}{\cos x-\sin x} - \int\left\{\frac{2}{(\cos x+\sin x)(\cos x-\sin x)}\right\}\frac{\sin 2x}{2}\,dx$$

$$= \sin x\cdot\cos x\cdot\log\frac{\cos x+\sin x}{\cos x-\sin x} - \int\frac{2}{\cos^2 x-\sin^2 x}\cdot\frac{\sin 2x}{2}\,dx$$

$$= \sin x\cdot\cos x\cdot\log\frac{\cos x+\sin x}{\cos x-\sin x} - \int\frac{\cancel{2}}{\cos 2x}\cdot\frac{\sin 2x}{\cancel{2}}\,dx$$

$$= \sin x\cdot\cos x\cdot\log\frac{\cos x+\sin x}{\cos x-\sin x} - \int\tan 2x\,dx$$

$$= \sin x\cdot\cos x\;\log\frac{\cos x+\sin x}{\cos x-\sin x} - \frac{1}{2}\log|\cos 2x| + c.$$

Theorem : $\int e^x [f(x) + f'(x)]dx = e^x f(x) + c.$

Solved Examples :

1) Given : $\int e^x (\tan x + 1) \sec x\, dx = e^x f(x) + c.$

Write $f(x)$ satisfying the above.

Soln : $\int e^x (\tan x + 1) \sec x\, dx = \int e^x (\sec x + \sec x \tan x) dx$

$\Rightarrow \quad e^x \sec x = e^x f(x) + c$

$\therefore \quad f(x) = \sec x.$

2) Integrate : $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2}\right) dx$

Soln : We know $f(x) = \tan^{-1} x \qquad f'(x) = \frac{1}{1+x^2}$

$\therefore \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2}\right) dx = \int e^x (f(x) + f'(x))\, dx$

$= e^x f(x) + c$

$\therefore \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2}\right) dx = e^x \tan^{-1} x + c.$

3) Integrate : $\int e^x \left(\frac{x^2+1}{(x+1)^2}\right) dx$

$I = \int e^x \left(\frac{x^2-1+2}{(x+1)^2}\right) dx = \int e^x \left[\frac{(x-1)(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}\right] dx$

$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$

$f(x) = \frac{x-1}{x+1} \qquad f'(x) = \frac{2}{(x+1)^2}$

$\therefore I = e^x \frac{x-1}{x+1} + c.$

4) Find : $\int \frac{2x-5}{(2x-3)^3} e^{2x} \cdot dx$

Sol^n: $I = \int \frac{2x-5}{(2x-3)^3} e^{2x} \cdot dx$

Put $2x = t$ $\quad 2dx = dt \Rightarrow dx = \frac{dt}{2}$

$\therefore I = \frac{1}{2}\int \frac{t-5}{(t-3)^3} e^{t} \cdot dt = \frac{1}{2}\int e^t \frac{(t-3)-2}{(t-3)^3} dt$

$= \frac{1}{2}\int e^t \left(\frac{1}{(t-3)^2} - \frac{2}{(t-3)^3}\right) dt$

If $f(t) = \frac{1}{(t-3)^2}$ then $f'(t) = \frac{-2}{(t-3)^3}$

$\therefore I = \frac{1}{2} e^t \frac{1}{(t-3)^2} + C = \frac{1}{2} \frac{e^{2x}}{(2x-3)^2} + c.$

5) Evaluate : $\int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$

Sol^n let $\log x = t$ i.e. $x = e^t$ $\quad dx = e^t dt$

$\therefore I = \int \left(\log t + \frac{1}{t^2}\right) e^t dt$

$= \int \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right) e^t dt$

$= \int e^t\left(\log t + \frac{1}{t}\right) dt + \int e^t\left(-\frac{1}{t} + \frac{1}{t^2}\right) dt$

If $f(t) = \log t$ $\quad f'(t) = \frac{1}{t}$ $\qquad$ if $f(t) = -\frac{1}{t}$ $\quad f'(t) = \frac{1}{t^2}$

$\therefore I = e^t \log t + e^t\left(-\frac{1}{t}\right) + C$ $\qquad e^{\log x} = x$

$= x \log(\log x) - x\left(\frac{1}{\log x}\right) + C$

$= x\left(\log(\log x) - \frac{1}{\log x}\right) + C.$

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

and $\int \frac{px+q}{ax^2+bx+c}$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$

Type i : $\int \frac{1}{\text{Quadratic Polynomial}} dx$ i.e. $\int \frac{1}{ax^2+bx+c} dx$

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right]$$

Thus the given integral depends upon the sign of (b^2-4ac).

Example: Evaluate $\int \frac{dx}{x^2+4x+7}$

$$x^2+4x+7 = \left[\left(x+\frac{4}{2}\right)^2-\left(\frac{16-4\cdot 7}{4}\right)\right]$$

$$= (x+2)^2+3$$

$$= (x+2)^2+(\sqrt{3})^2$$

$$I = \int \frac{dx}{(x+2)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+C$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C.\right]$$

Also $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C.$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$$

Type II : $\int \frac{\text{Linear Polynomial}}{\text{Quadratic Polynomial}}\,dx$ i.e. $\int \frac{px+q}{ax^2+bx+c}\,dx$ $p \neq 0$, $a \neq 0$.

Here $px+q = A(\text{Derivative of denominator}) + B$
$= A(2ax+b) + B$

comparing $p = 2aA$ $q = Ab + B$

Solving, we get A and B.

It breaks up into two integrals one of which is of the form $\int \frac{f'(x)}{f(x)}$ and the other is of the form

$$\int \frac{dx}{x^2+a^2}, \quad \int \frac{dx}{x^2-a^2} \quad \text{or} \quad \int \frac{dx}{a^2-x^2}$$

Same is true for $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$ break into two parts.

Example $I = \int \frac{x+2}{2x^2+6x+5}\,dx$

$$I = \int \frac{x+2}{2x^2+6x+5}\,dx \qquad \frac{d}{dx}(2x^2+6x+5) = 4x+6$$

Let $(x+2) = A(4x+6) + B = 4Ax + 6A + B$

$4A = 1$ $\quad 6A + B = 2$ $\quad \Rightarrow 6 \cdot \frac{1}{4} + B = 2$

$A = \frac{1}{4}$ $\quad \Rightarrow B = \frac{1}{2}$

$\therefore x+2 = \frac{1}{4}(4x+6) + \frac{1}{2}$

$$\therefore I = \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5} + \frac{1}{2}\int \frac{dx}{2x^2+6x+5} = \frac{1}{4}I_1 + \frac{1}{2}I_2$$

$$I_1 = \int \frac{4x+6}{2x^2+6x+5}\,dx \quad \text{of the form} \quad \int \frac{f'(x)}{f(x)}\,dx$$

$$\therefore I_1 = \log|2x^2+6x+5|$$

$$I_2 = \int \frac{dx}{2x^2+6x+5} \qquad 2x^2+6x+5 = 2\left(x^2+3x+\frac{5}{2}\right)$$

$$= 2\left[\left(x+\frac{3}{2}\right)^2 - \frac{9-4\cdot\frac{2\times5}{2}}{4}\right]$$

$$= 2\left[\left(x+\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$$

$$\therefore I_2 = \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{2}}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right) = \tan^{-1}(2x+3)$$

$$\therefore I = \frac{1}{4}\left(\log|2x^2+6x+5|\right) + \frac{1}{2}\tan^{-1}(2x+3) + C.$$

Solved Examples

Remember: (i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$ (iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C$ (v) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C$ (vii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$.

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1) Evaluate: $\int \frac{dx}{2x^2+7x+13}$

Soln: $I = \int \frac{dx}{2x^2+7x+13}$

$$2x^2+7x+13 = 2\left[x^2+\frac{7}{2}x+\frac{13}{2}\right]$$

$$\left[\because ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]\right]$$

$$\therefore 2x^2+7x+13 = 2\left[\left(x+\frac{7}{2\cdot 2}\right)^2 - \left(\frac{(7)^2-4\cdot 2\cdot 13}{4\cdot (2)^2}\right)\right]$$

$$= 2\left[\left(x+\frac{7}{4}\right)^2 - \left(\frac{49-104}{16}\right)\right]$$

$$= 2\left[\left(x+\frac{7}{4}\right)^2 + \frac{55}{16}\right]$$

$$= 2\left[\left(x+\frac{7}{4}\right)^2 + \left(\frac{\sqrt{55}}{4}\right)^2\right]$$

$$\therefore \quad I = \frac{1}{2}\int \frac{dx}{\left(x+\frac{7}{4}\right)^2+\left(\frac{\sqrt{55}}{4}\right)^2} = \frac{1}{2}\times\frac{1}{\frac{\sqrt{55}}{4}}\tan^{-1}\left(\frac{x+\frac{7}{4}}{\frac{\sqrt{55}}{4}}\right)+c$$

$$= \frac{2}{\sqrt{55}}\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right)+c$$

2) Evaluate : $\int \frac{dx}{1-6x-9x^2}$

$$1-6x-9x^2 = \frac{1}{9}\left(\frac{1}{9}-\frac{2}{3}x-x^2\right) = \frac{1}{9}\left[\frac{1}{9}-\left(x^2+\frac{2}{3}x\right)\right]$$

$$= \frac{1}{9}\left[\frac{1}{9}-\left(x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\frac{1}{9}\right)\right] = \frac{1}{9}\left[\frac{1}{9}-\left[\left(x+\frac{1}{3}\right)^2-\frac{1}{9}\right]\right]$$

$$I = \frac{1}{9}\int \frac{dx}{\frac{1}{9}-\left\{\left(x+\frac{1}{3}\right)^2-\frac{1}{9}\right\}} = \frac{1}{9}\int \frac{dx}{\left(\frac{\sqrt{2}}{3}\right)^2-\left(x+\frac{1}{3}\right)^2}$$

$$I = \frac{1}{9}\,\frac{1}{2\cdot\frac{\sqrt{2}}{3}}\cdot\log\frac{\frac{\sqrt{2}}{3}+x+\frac{1}{3}}{\frac{\sqrt{2}}{3}-x-\frac{1}{3}}+c = \frac{1}{6\sqrt{2}}\log\left|\frac{\sqrt{2}+(3x+1)}{\sqrt{2}-(3x+1)}\right|+c.$$

Problems related with integrals reducible to form $\int \frac{dx}{ax^2+bx+c}$

3) Evaluate : (i) $\int \frac{x}{x^4+x^2+1}dx$ (ii) $\int \frac{\sec^2 x}{5\tan^2 x-12x+4}dx$.

(iii) $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$ (iv) $\int \frac{e^x}{e^{2x}+6e^x+5}dx$

Ans (i) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$ (ii) $\frac{1}{\sqrt{34}}\tan^{-1}\left(\frac{5\tan x-6}{\sqrt{34}}\right)+c$

(iii) $\tan^{-1}(\sin x+2)+c$ (iv) $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c.$

4) Solved problems with the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(i) $\int \frac{dx}{\sqrt{9+8x-x^2}}$ (ii) $\int \frac{dx}{\sqrt{x}\sqrt{(5-x)}}$ (iii) $\int \frac{dx}{\sqrt{8+3x-x^2}}$

(iv) $\int \frac{dx}{\sqrt{(x-1)(x-2)}}$

Ans (i) Reduce to from $\int \frac{dx}{\sqrt{a^2-x^2}}$ $I = \sin^{-1}\left(\frac{x-4}{5}\right)+C$

(ii) $\sin^{-1}\left(\frac{2x-5}{5}\right)+C$ (iii) $\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

(iv) Reduce to from $\int \frac{dx}{\sqrt{x^2-c^2}}$ $I = \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right)+C$.

5) Problems related with integrals reducible to form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(i) $\int \frac{2x}{\sqrt{1-x^2-x^4}}$ (ii) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}$

(iii) $\int \frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}dx$ (iv) $\int \frac{\sin x\cdot\cos x}{\sqrt{9-4\sin^4 x}}\cdot dx$

Ans (i) $\sin^{-1}\frac{2x^2+1}{\sqrt{5}}+C$ (ii) $\sin^{-1}\left(\frac{e^x+2}{3}\right)+C$

(iii) $\log\left[(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right]+C$

(iv) $\frac{1}{4}\sin^{-1}\left(\frac{2\sin^2 x}{3}\right)+C$

6) Integrate : $\int \frac{x+3}{x^2-2x-5}\,dx$.

Sol^n We have $\frac{d(x^2-2x-5)}{dx} = 2x-2$

Let $x+3 = A(2x-2)+B \Rightarrow 2Ax - 2A + B$

$= 2Ax + (B-2A)$

$\therefore\ 2A = 1 \qquad B-2A = 3$

$A = \frac{1}{2} \qquad B - 2\cdot\frac{1}{2} = 3 \Rightarrow B = 4$

$\therefore\ x+3 = \frac{1}{2}(2x-2) + 4$

$\therefore\ I = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}\,dx + 4\int \frac{dx}{x^2-2x-5}$

$I = \frac{1}{2} I_1 + 4 I_2$

$I_1 = \int \frac{2x-2}{x^2-2x-5}\,dx = \log|x^2-2x-5|$

$I_2 = \int \frac{dx}{x^2-2x-5} \qquad ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$

$\therefore\ x^2-2x-5 = \left(x + \frac{-2}{2}\right)^2 - \left(\frac{(-2)^2-4\cdot1\cdot(-5)}{4}\right) = (x-1)^2 - \left(\frac{4+20}{4}\right)$

$= (x-1)^2 - (\sqrt{6})^2$

$\therefore\ \int \frac{dx}{x^2-2x-5} = \int \frac{dx}{(x-1)^2-(\sqrt{6})^2} = \frac{1}{2\sqrt{6}} \log\left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + c.$

$\therefore\ I = \frac{1}{2}\log|x^2-2x-5| + \frac{2}{\sqrt{6}}\log\left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + c.$

7) Evaluate : (i) $\int \frac{4x+1}{\sqrt{2x^2+x-3}}$ (ii) $\int \frac{x+3}{\sqrt{5-4x-x^2}}$ (iii) $\int \frac{x+2}{\sqrt{x^2+2x-1}} dx$

Soln: (i) $I = \int \frac{4x+1}{\sqrt{2x^2+x-3}}$

$\frac{d(2x^2+x-3)}{dx} = 4x+1$ $\therefore$ put $2x^2+x-3 = t$

$\therefore (4x+1)dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt = \frac{t^{\frac{1}{2}}}{1/2} + C = 2\sqrt{t} + C = 2\sqrt{2x^2+x-3} + C$

(ii) $I = \int \frac{x+3}{\sqrt{5-4x-x^2}} dx$

$\frac{d(5-4x-x^2)}{dx} = -4-2x$

Let $x+3 = A(-4-2x) + B = -4A - 2Ax + B$

$x+3 = -2Ax + B - 4A$

$-2A = 1 \Rightarrow A = \frac{-1}{2}$ $B - 4\left(\frac{-1}{2}\right) = 3$

$B + 2 = 3 \Rightarrow B = 1$

$\therefore x+3 = \frac{-1}{2}(-4-2x) + 1$

$\therefore I = \frac{-1}{2}\int \frac{(-4-2x)}{\sqrt{5-4x-x^2}} dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$

$I = \frac{-1}{2} I_1 + I_2$

$I_1 = \int \frac{(-4-2x)}{\sqrt{5-4x-x^2}} dx$ put $5-4x-x^2 = t$

$(-4-2x)dx = dt$

$\therefore I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{5-4x-x^2}$

$5-4x-x^2 = 5-(x^2+4x) = 5-(x^2+4x+4-4)$

$= 9-(x^2+4x+4)$

$= 9-(x+2)^2 = (3)^2-(x+2)^2$

$$I_2 = \int \frac{dx}{\sqrt{(3)^2-(x+2)^2}} = \sin^{-1}\left(\frac{x+2}{3}\right) + C.$$

$$\therefore I = -\frac{1}{2} I_1 + I_2 = \frac{-1}{2}\, 2\sqrt{5-4x-x^2} + \sin^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\therefore I = -\sqrt{5-4x-x^2} + \sin^{-1}\left(\frac{x+2}{3}\right) + C.$$

(iii) $$I = \int \frac{x+2}{\sqrt{x^2+2x-1}}\, dx$$

Ans : $\sqrt{x^2+2x-1} + \log\left|(x+1)+\sqrt{x^2+2x-1}\right| + C.$

8) Evaluate : $$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\, dx$$

Soln : $(x-5)(x-4) = x^2-(5+4)x+5\times 4 = x^2-9x+20$

$$\frac{d(x^2-9x+20)}{dx} = 2x-9$$

let $6x+7 = A(2x-9)+B$

$= 2Ax+(B-9A)$

$\therefore 6 = 2A \quad A = 3 \qquad B-9A = 7 \Rightarrow B = 34$

$A = 3$ & $B = 34$

$\therefore 6x+7 = 3(2x-9)+34$

$$\therefore I = 3\int \frac{(2x-9)}{\sqrt{x^2-9x+20}} + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$$

$I = 3I_1 + 34 I_2$

$$I_1 = \int \frac{(2x-9)}{\sqrt{x^2-9x+20}}\, dx = 2\sqrt{x^2-9x+20} + C_1$$

[$\because$ by putting $x^2-9x+20 = t \quad (2x-9)\,dx = dt$]

$$I_2 = \int \frac{dx}{\sqrt{x^2-9x+20}}$$

We know $\quad ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$

$$\therefore x^2-9x+20 = \left(x-\frac{9}{2}\right)^2 - \left(\frac{81-4\cdot 1\cdot 20}{4}\right)$$

$$= \left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{4}\right)$$

$$= \left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\therefore I_2 = \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}} = \log\left|\left(x-\frac{9}{2}\right)\ \sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|$$

$$= \log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|$$

$$\therefore I = 3\cdot 2\sqrt{x^2-9x+20} + 34\cdot\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| + C$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| + C.$$

9) Evaluate : $\displaystyle\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}\, d\theta$.

Sol^n: $\displaystyle I = \int \frac{(3\sin\theta-2)\cos\theta}{5-(1-\sin^2\theta)-4\sin\theta}\cdot d\theta = \int \frac{(3\sin\theta-2)\cos\theta}{5-1+\sin^2\theta-4\sin\theta}\cdot d\theta$

$$I = \int \frac{(3\sin\theta-2)\cos\theta}{\sin^2\theta-4\sin\theta+4}\, d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta\cdot d\theta = dt$

$$\therefore I = \int \frac{(3t-2)\, dt}{t^2-4t+4}$$

$$I = \int \frac{(3t-2)\,dt}{t^2-4t+4}$$

$$\frac{d(t^2-4t+4)}{dt} = 2t-4$$

Let $3t-2 = A(2t-4)+B$

$= 2At + B - 4A$

$2A = 3 \Rightarrow A = \frac{3}{2}$ $\qquad B - \cancel{4}^{2}\cdot\frac{3}{\cancel{2}} = -2$

$B = -2+6 = 4$

$\therefore A = \frac{3}{2}$ & $B = 4$.

$\therefore 3t-2 = \frac{3}{2}(2t-4)+4.$

$$\therefore I = \frac{3}{2}\int \frac{(2t-4)}{t^2-4t+4}\,dt + 4\int \frac{dt}{t^2-4t+4}.$$

$t^2-4t+4 = t^2-2t-2t+4 = t(t-2)-2(t-2) = (t-2)(t-2)$

$= (t-2)^2$

$\therefore I = \frac{3}{2}\log|t^2-4t+4| + 4\int (t-2)^{-2}\,dt$

$= \frac{3}{2}\log|t^2-4t+4| + 4\left(\frac{-1}{t-2}\right) + C$

$\therefore = \frac{3}{2}\log|\sin^2\theta - 4\sin\theta+4| - \frac{4}{\sin\theta-2} + C$

$= \frac{3}{2}\log|\sin\theta-2|^2 - \frac{4}{\sin\theta-2} + C$

$= 3\log|\sin\theta-2| - \frac{4}{\sin\theta-2} + C.$

10) Evaluate : $\int \frac{x^2+5x+3}{x^2+3x+2}\, dx$.

$$I = \int \frac{(x^2+3x+2)+(2x+1)}{x^2+3x+2}\, dx = \int \left(1+ \frac{2x+1}{x^2+3x+2}\right) dx$$

$$= x + \int \frac{2x+1}{x^2+3x+2}\, dx = x + \int \frac{2x+3}{x^2+3x+2} - 2\int \frac{dx}{x^2+3x+2}$$

$$I = x + \log|x^2+3x+2| - 2\int \frac{dx}{x^2+3x+2}$$

We know $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$

$$\therefore x^2+3x+2 = \left(x+\frac{3}{2}\right)^2 - \left(\frac{9-4.1.2}{4}\right)$$

$$= \left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\therefore I = x + \log|x^2+3x+2| - 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= x + \log|x^2+3x+2| - 2\frac{1}{2\cdot\frac{1}{2}} \log\left|\frac{\left(x+\frac{3}{2}\right)-\frac{1}{2}}{\left(x+\frac{3}{2}\right)+\frac{1}{2}}\right| + c$$

$$= x + \log|x^2+3x+2) - 2\log\left|\frac{x+1}{x+2}\right| + c.$$

Integration of a Rational Function of the Form

(i) $\int \frac{x^2+a^2}{x^4+kx^2+a^4}\,dx$ (ii) $\int \frac{x^2-a^2}{x^4+kx^2+a^4}\,dx$

where k is a constant, positive, negative or zero.

To evaluate these type of integrals, we proceed as follows:

(i) $I = \int \frac{x^2+a^2}{x^4+kx^2+a^4}\,dx$

Divide numerator and denominator by x^2, we get

$$I = \int \frac{1+\frac{a^2}{x^2}}{x^2+k+\frac{a^4}{x^2}}\,dx$$

Put $x-\frac{a^2}{x}=t$

$\left(1+\frac{a^2}{x^2}\right)dx = dt$

Also $\left(x-\frac{a^2}{x}\right)^2 = t^2 \Rightarrow x^2+\frac{a^4}{x^2}-2a^2 = t^2$

$x^2+\frac{a^4}{x^2} = t^2+2a^2$

$I = \int \frac{dt}{t^2+2a^2+k}$ which is of the form $\int \frac{dt}{t^2+c^2}$ or $\int \frac{dt}{t^2-c^2}$

This can be easily integrated.

(ii) We can evaluate integrals of the form $\int \frac{x^2-a^2}{x^4+kx^2+a^4}\,dx$

in similar way as given in above form.

Here we put $x+\frac{a^2}{x}=t$.

Remember

(1) $$\int \frac{x^2}{x^4+kx^2+a^4} = \frac{1}{2}\int \frac{2x^2}{x^4+kx^2+a^4}\,dx = \frac{1}{2}\int \frac{(x^2+a^2)+(x^2-a^2)}{x^4+kx^2+a^4}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+a^2}{x^4+kx^2+a^2}\,dx + \frac{1}{2}\int \frac{x^2-a^2}{x^4+kx^2+a^4}$$

(2) $$\int \frac{dx}{x^4+kx^2+a^4} = \frac{1}{2a^2}\int \frac{2a^2}{x^4+kx^2+a^4}\,dx$$

$$= \frac{1}{2a^2}\int \frac{x^2+a^2}{x^4+kx^2+a^4}\,dx - \frac{1}{2a^2}\int \frac{x^2-a^2}{x^4+kx^2+a^4}\,dx.$$

Solved Examples

(1) Evaluate : (i) $\int \frac{x^2+1}{x^4+1}\,dx$ (ii) $\int \frac{x^2-1}{x^4+1}\,dx$ (iii) $\int \frac{x^2-1}{x^4+x^2+1}\,dx$

Sol^n : (i) $I = \int \frac{x^2+1}{x^4+1}$

Dividing numerator and denominator by x^2

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$(a-b)^2 = a^2+b^2-2ab$

$\therefore x^2+\frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2+2$

$$I = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx$$

put $x-\frac{1}{x} = t$

$\left(1+\frac{1}{x^2}\right)dx = dt$

$$\therefore I = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C \qquad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+C.$$

(ii) $I = \int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$

put $x+\frac{1}{x} = t \qquad \left(1-\frac{1}{x^2}\right)dx = dt$

$\therefore \; I = \int \frac{dt}{t^2-(\sqrt{2})^2}$ We know $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore \; I = \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$

(iii) $I = \int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}\,dx$

$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1}\,dx$

$x+\frac{1}{x} = t \qquad \left(1-\frac{1}{x^2}\right)dx = dt$

$\therefore \; I = \int \frac{dt}{t^2-(1)^2} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

$= \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C.$

5) Evaluate: $\int \frac{dx}{\sin^6 x + \cos^6 x}$

Sol^n : $I = \int \frac{dx}{\sin^6 x + \cos^6 x}$

Dividing num and deno by $\cos^6 x$.

$$I = \int \frac{\sec^6 x}{\tan^6 x + 1}\, dx = \int \frac{\sec^4 x \cdot \sec^2 x \cdot dx}{1 + \tan^6 x}$$

$$= \int \frac{(1+\tan^2 x)^2 \sec^2 x}{1 + \tan^6 x} \cdot dx$$

put $\tan x = t$ $\quad \sec^2 x \cdot dx = dt$

$$I = \int \frac{(1+t^2)^2 dt}{1+t^6} = \int \frac{(1+t^2)^2}{1+(t^2)^3} \cdot dt$$

$\because a^3+b^3 = (a+b)(a^2-ab+b^2)$

$\therefore 1+(t^2)^3 = (1+t^2)(t^4+1-t^2)$

$$\therefore I = \int \frac{(1+t^2)^{\cancel{2}}}{\cancel{(1+t^2)}(t^4-t^2+1)} \cdot dt = \int \frac{1+t^2}{t^4-t^2+1} \cdot dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2-1+\frac{1}{t^2}} \cdot dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+1} \cdot dt$$

Put $t - \frac{1}{t} = z$ $\quad \left(1+\frac{1}{t^2}\right)dt = dz$

Ans. $I = \tan^{-1}(\tan x - \cot x) + C$.

6) Evaluate $\int \frac{dx}{x^4+x^2+1}$

Sol^n:

$$I = \int \frac{dx}{x^4+x^2+1} = \frac{1}{2}\int \frac{2dx}{x^4+x^2+1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx - \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2} I_1 - \frac{1}{2} I_2$$

$$I_1 = \int \frac{x^2+1}{x^4+x^2+1}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)$$

$$I_2 = \int \frac{x^2-1}{x^4+x^2+1}\,dx = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|$$

$$\therefore I = \frac{1}{2\cdot\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) + \frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C.$$

11) Evaluate $\int \frac{x^2}{x^4+x^2+1}\,dx$

Sol^n $$I = \frac{1}{2}\int \frac{2x^2}{x^4+x^2+1}\,dx = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) + \frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C.$$

12) Evaluate $\int \sqrt{\tan x} \cdot dx$.

Sol^n: $I = \int \sqrt{\tan x} \cdot dx$

put $\tan x = y^2$ $\quad \sec^2 x \cdot dx = 2y \cdot dy$

$$dx = \frac{2y \cdot dy}{\sec^2 x} = \frac{2y \cdot dy}{1+\tan^2 x} = \frac{2y\,dy}{1+y^4}$$

$$\therefore I = \int \frac{y \cdot 2y \cdot dy}{1+y^4} = \int \frac{2y^2}{y^4+1}\,dy.$$

$$= \int \frac{y^2+1}{y^4+1}\,dy + \int \frac{y^2-1}{y^4+1} \cdot dy$$

$$I = I_1 + I_2$$

$$I_1 = \int \frac{y^2+1}{y^4+1} \cdot dy = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y^2-1}{y\sqrt{2}}\right)$$

$$I_2 = \int \frac{y^2-1}{y^4+1} \cdot dy = \frac{1}{2\sqrt{2}} \log\left|\frac{y^2-\sqrt{2}y+1}{y^2+\sqrt{2}y+1}\right|$$

$$I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right| + c.$$

13) Evaluate: $\int \sqrt{\cot x} \cdot dx$

Sol^n: $I = \int \sqrt{\cot x} \cdot dx$ $\quad \sqrt{\cot x} = t$ $\quad \cot x = t^2$

$-\operatorname{cosec}^2 x\, dx = dt$ $\quad$ (same method as above)

ANS: $$\frac{-1}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}} \log\left|\frac{\cot x - \sqrt{2\cot x} + 1}{\cot x + \sqrt{2\cot x} + 1}\right| + c.$$

Integrals of the type $\int \frac{dx}{a+b\cos x}$ and $\int \frac{dx}{a+b\sin x}$

Integration of rational function of $\cos x$ and $\sin x$ is simplied by the substitution $\tan\frac{x}{2} = t$, which converts the integral into integral of a rational function t.

Put $\tan\frac{x}{2} = t$ $\qquad \sec^2\frac{x}{2} \cdot \frac{1}{2} dx = dt$

$$dx = \frac{2\,dt}{\sec^2\frac{x}{2}} = \frac{2\cdot dt}{1+\sec^2\frac{x}{2}} = \frac{2\cdot dt}{1+t^2}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$I = \int \frac{dx}{a+b\cos x}$$

$$= \int \frac{1}{a+b\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2t}{1+t^2} \cdot dt = \int \frac{2\cdot dt}{a(1+t^2)+b(1-t^2)}$$

$$= \int \frac{2\,dt}{(a+b)+(a-b)t^2}$$

Case i $\quad a > b$, then $(a-b)$ is positive.

$$I = \frac{2}{a-b}\int \frac{dt}{t^2+\left(\frac{a+b}{a-b}\right)} = \frac{2}{(a-b)}\int \frac{dt}{t^2+\left(\sqrt{\frac{a+b}{a-b}}\right)^2}$$

Integral of the form $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$\therefore \quad I = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2}\right) + C.$$

Case II : let $a < b$ so then $\frac{a+b}{a-b}$ is negative.

This integral is of the form $\int \frac{dy}{c^2-y^2}$ or $\int \frac{dx}{a^2-x^2}$

$$I = \frac{2}{b-a} \int \frac{dt}{\left(\sqrt{\frac{b+a}{b-a}}\right)^2 - t^2} = \frac{1}{\sqrt{b^2-a^2}} \log \left| \frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\frac{x}{2}} \right| + c.$$

Case III : When $a = b$

then $I = \int \frac{dx}{b+\cos x} = \int \frac{dx}{a + a\cos x} = \frac{1}{a} \int \frac{dx}{1+\cos x} = \frac{1}{a} \int \frac{dx}{2\cos^2\frac{x}{2}}$

$$= \frac{1}{2a} \int \sec^2\frac{x}{2}\, dx = \frac{1}{2a} \cdot \frac{\tan\frac{x}{2}}{1/2} = \frac{1}{a} \tan\frac{x}{2}$$

$$\therefore I = \int \frac{dx}{b + a\cos x} = \frac{1}{a} \tan\frac{x}{2}.$$

Let us evaluate $I = \int \frac{dx}{a + b\sin x}$ a & b are positive.

Put $\tan\frac{x}{2} = t$ $\quad \sin x = \frac{2t}{1+t^2}$ $\quad dx = \frac{2\cdot dt}{1+t^2}$

$$I = \int \frac{1}{a + b\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2\cdot dt}{a(1+t^2) + 2b.t}$$

$$= \frac{2}{a} \int \frac{dt}{t^2 + \frac{2b}{a}t + 1} = \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2 + 1 - \frac{b^2}{a^2}}$$

$$= \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2 + \frac{a^2-b^2}{a^2}}$$

Case I: If $a^2 > b^2$, then a^2-b^2 is positive.

$$I = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 + \left(\frac{\sqrt{a^2-b^2}}{a}\right)^2} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right) + c$$

Case II: $a^2 < b^2$ then (b^2-a^2) is positive.

$$I = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 - \frac{b^2-a^2}{a}} = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2 - \left(\frac{\sqrt{b^2-a^2}}{a}\right)^2}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right| + c.$$

Case III: When $a=b$.

$$I = \int \frac{dx}{a+a\sin x} = \frac{1}{a}\int \frac{dx}{1+\sin x} = \frac{1}{a}\int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2\cdot dt}{1+t^2}$$

$$= \frac{1}{a}\int \frac{2\,dt}{t^2+2t+1} = \frac{1}{a}\int \frac{(t^2+1)-(t^2-1)}{t^2+2t+1}\,dt$$

$$I = \frac{1}{a}\int \frac{t^2+1}{t^2+2t+1}\,dt - \frac{1}{a}\int \frac{t^2-1}{t^2+2t+1}\,dt \quad \ldots$$

Integrals of the type $\int \frac{dx}{a+b\cos x+c\sin x}$.

Put $\tan\frac{x}{2} = t$ then $dx = \frac{2\,dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$ $\quad \sin x = \frac{2t}{1+t^2}$ $\quad \ldots$

Integrals of the form $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\,dx$

Numerator $=$ A (Denominator) $+$ B $\frac{d(\text{Denominator})}{dx}$

$\therefore p\cos x + q\sin x + r = A(a\cos x + b\sin x + c) + B(-a\sin x + b\cos x) + C$

Now, equating coefficients of $\cos x$, $\sin x$ and constant terms on both sides, we get

$p = Aa + Bb$
$q = Ab - Ba$
$r = Ac + C$

Solving $A = \frac{pa+qb}{a^2+b^2}$, $B = \frac{pb-qa}{a^2+b^2}$, $C = r - \frac{(pa+qb)c}{a^2+b^2}$

$\therefore I = A\int dx + B\int \frac{-a\sin x + b\cos x}{a\cos x + b\sin x + c}\,dx + C\int \frac{dx}{a\cos x + b\sin x + c}$

Hence $I = Ax + B\log|a\cos x + b\sin x + c| + C\int \frac{dx}{a\cos x + b\sin x + c}$

Example : (i) $\int \frac{26\sin x - 13\cos x}{4\sin x - 7\cos x}\,dx$ (ii) $\int \frac{\sin x + 8\cos x}{2\sin x + 3\cos x}\,dx$

Soln: (i) $I = \int \frac{26\sin x - 13\cos x}{4\sin x - 7\cos x}\,dx$

Let $26\sin x - 13\cos x = A(4\sin x - 7\cos x) + B(4\cos x + 7\sin x)$

$= 4A\sin x - 7A\cos x + 4B\cos x + 7B\sin x$

$= (4A + 7B)\sin x + (4B - 7A)\cos x$

$\therefore 4A + 7B = 26$, $-7A + 4B = -13$ Solving $A = 3$, $B = 2$

$I = 3\int dx + 2\int \frac{4\cos x + 7\sin x}{4\sin x - 7\cos x}\,dx$

$= 3x + 2\log|4\sin x - 7\cos x| + C.$

(ii) $\quad I = \int \frac{\sin x + 8\cos x}{2\sin x + 3\cos x} \cdot dx$

let $\quad \sin x + 8\cos x = A(2\sin x + 3\cos x) + B(2\cos x - 3\sin x)$

$$= 2A\sin x + 3A\cos x + 2B\cos x - 3B\sin x$$

$$= (2A - 3B)\sin x + (3A + 2B)\cos x$$

$\therefore \quad 2A - 3B = 1 \qquad \frac{A}{26} = \frac{B}{13} = \frac{1}{13} \Rightarrow A = \frac{26}{13} \qquad B = 1$

$\quad\;\; 3A + 2B = 8$

$\therefore \; \sin x + 8\cos x = 2(2\sin x + 3\cos x) + (2\cos x - 3\sin x)$

$$I = 2\int dx + \int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}$$

$$= 2x + \log|2\sin x + 3\cos x| + C.$$

Integration by Partial Fractions

In general, rational functions can be quite difficult to integrate.

Rational Function: An expression of the form $\frac{f(x)}{\phi(x)}$.

For the sake of guidance, the following notes are given:

(a) When the degree of the numerator of a rational algebraic function is equal or greater than that of the denominator, then divide the numerator by denominator till the degree of the remainder is less than that of the denominator.

Thus, a given fraction = a polynomial + a proper rational algebraic fraction.

For example, : $\frac{x^4+5x^3-7x^2+9x+3}{x^3-x^2+4x+7} = (x+6) + \frac{-5x^2-22x-29}{x^3-x^2+4x+7}$.

(b) Firstly, we resolve the denominator into real factors or quadratic, repeated or non-repeated, then we resolve into partial fraction in accordance with the following rules:

Rule 1. Corresponding to a <u>non-repeated linear factor</u> $(x-\alpha)$ in the denominator, there is a partial fraction of the type: $\frac{A}{x-\alpha}$

Put the non-repeated linear factor $x-\alpha=0$. Thus $x=\alpha$.

Put $x=\alpha$ everywhere in the given fraction except in the factor $(x-\alpha)$ itself.

Rule 2 : corresponding to repeated linear factor $(x-\alpha)^r$ in the denominator, there are r partial fractions.

$$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3} + \cdots + \frac{A_r}{(x-\alpha)^r}$$

Rule 3 : Corresponding to a non-repeated quadratic factors $(x^2+\alpha x+\beta)$ in the denominator, there is a partial fraction of type : $\frac{Ax+B}{(x^2+\alpha x+\beta)}$

Rule 4 : corresponding to a repeated quadratic factor $(x^2+\alpha x+\beta)^r$ in the denominator, there are r partial fractions of the type :

$$\frac{A_1x+B_1}{x^2+\alpha x+\beta} + \frac{A_2x+B_2}{(x^2+\alpha x+\beta)^2} + \cdots + \frac{A_rx+B_r}{(x^2+\alpha x+\beta)^r}$$

Rule 5 : If there are only even powers of x put $x^2=y$, then do as above.

We indicate the type of simpler partial fractions which are to be associated with various kinds of rational function in the form of following Table :

S. No.	Form of Rational Function	Form of Partial Fractions
1.	$\frac{px+q}{(x-\alpha)(x-\beta)}$; $\alpha \neq \beta$	$\frac{A}{x-\alpha}+\frac{B}{x-\beta}$
2.	$\frac{px+q}{(x-\alpha)^2}$	$\frac{A}{x-\alpha}+\frac{B}{(x-\alpha)^2}$
3.	$\frac{px^2+qx+r}{(x-\alpha)(x-\beta)(x-\gamma)}$	$\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{x-\gamma}$
4.	$\frac{px^2+qx+r}{(x-\alpha)^2(x-\beta)}$	$\frac{A}{x-\alpha}+\frac{B}{(x-\alpha)^2}+\frac{C}{x-\beta}$
5.	$\frac{px^2+qx+r}{(x-\alpha)(x^2+\beta x+\gamma)}$, where $x^2+\beta x+\gamma$ cannot be factorised further.	$\frac{A}{x-\alpha}+\frac{Bx+C}{x^2+\beta x+\gamma}$

Type I. When denominator contains only non-repeated linear factors.

(1) Evaluate the following :

(i) $\int \frac{1}{(x+1)(x+2)}\,dx$ (ii) $\int \frac{dx}{x^2-9}$ (iii) $\int \frac{2x}{x^2+3x+2}\,.dx$

(iv) $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$ (v) $\int \frac{1}{x-x^3}\,dx$ (vi) $\int \frac{5x}{(x+1)(x^2-4)}\,.dx$.

Sol^n (i) $I = \int \frac{1}{(x+1)(x+2)}\,dx$

Let $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

Multiplying both side by $(x+1)(x+2)$

$1 = A(x+2) + B(x+1)$ put $x=-1$ $A(-1+2) = 1$

$A = 1$

put $x = -2$ $B(-2+1) = 1$

$B = -1$

$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} - \frac{1}{x+2}$

$\therefore I = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \log|x+1| - \log|x+2| + C.$

(ii) $I = \int \frac{dx}{x^2-9} = \int \frac{dx}{(x+3)(x-3)}$

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$ $\Rightarrow A(x-3) + B(x+3) = 1$

$(A+B)x + (3A-3B) = 0.x + 1$

$A+B=0$ $3A-3B=1$

$A = 1/6$ $B = -1/6$

$I = \frac{1}{6}\int \frac{dx}{(x+3)} - \frac{1}{6}\int \frac{dx}{x-3}$

$= \frac{1}{6}\log|x+3| - \frac{1}{6}\log|x-3| + C$ $= \frac{1}{6}\log\left|\frac{x+3}{x-3}\right| + c$

(iii) $I = \int \frac{2x}{x^2+3x+2} \cdot dx$

$x^2+3x+2 = x^2+2x+x+2 = x(x+2)+1(x+2) = (x+1)(x+2)$

$\therefore \quad I = \int \frac{2x}{(x+1)(x+2)} dx$

let $\frac{2x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$A(x+2) + B(x+1) = 2x$

Putting $x = -1, -2$ $\quad A(-1+2) = 2(-1)$ $\quad B(-2+2)+B(-2+1) = -4$

$A = -2$ $\quad -B = -4 \Rightarrow B = 4$

$\therefore \quad A = -2, \quad B = 4$

$\therefore \quad \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$

$I = -2\int \frac{dx}{(x+1)} + 4\int \frac{dx}{(x+2)} = -2\log|x+1| + 4\log|x+2| + C.$

(iv) $I = \int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$

let $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$

$A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) = 2x-1$

put $x = 1$, $\quad A(1+2)(1-3) = 2\times1-1 \Rightarrow -6A = 1 \quad A = \frac{-1}{6}$

put $x = -2$ $\quad B(-2-1)(-2-3) = 2(-2)-1$

$B\times -3\times -5 = -4-1 \quad ; \quad B = -\frac{1}{3}$

put x 3 $\quad C(2)(5) = 5 \quad C = \frac{1}{2}$

$$\therefore \quad I = \frac{-1}{6}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} + \frac{1}{2}\int \frac{dx}{x-3}$$

$$\therefore \quad I = -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C.$$

(v) $$I = \int \frac{1}{x-x^3}\,dx = \int \frac{1}{x(1-x^2)}\,dx = \int \frac{1}{x(1-x)(1+x)}\,dx.$$

let $$\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$\Rightarrow \quad A(1-x)(1+x) + B(x)(1+x) + C(x)(1-x) = 1$$

put $x = 0$ $\quad A(\cancel{0}-\cancel{0})(0+1) = 1 \quad \Rightarrow \quad A = 1$

put $x = 1$ $\quad B(1)(1+1) = 1 \quad B = \frac{1}{2}$

put $x = -1$ $\quad C(-1)(-2) = 1 \quad C = -\frac{1}{2}$

$$\therefore \quad I = \int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{1-x} - \int \frac{dx}{1+x}$$

$$= \log|x| + \frac{1}{2}\log|1-x| - \log|1+x| + C$$

$$= \log|x| - \frac{1}{2}\{\log|1+x| + \log|1-x|\} + C$$

$$= \log|x| - \frac{1}{2}\{\log(1+x)(1-x)\} + C$$

$$= \log|x| - \frac{1}{2}\log(1-x^2) + C$$

$$= \log|x| - \log(1-x^2)^{1/2}$$

$$= \log\frac{x}{\sqrt{1-x^2}} + C.$$

(vi) $I = \int \frac{5x}{(x+1)(x^2-2^2)} = \int \frac{5x}{(x+1)(x+2)(x-2)}\,dx$

let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$\Rightarrow A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) = 5x$

putting $x = -1, -2, 2,$

$A(-1+2)(-1-2) = -5$; $A(1)(-3) = -5$; $A = \frac{5}{3}$

$B(-2+1)(-2-2) = -10$; $B(-1)(-4) = -10$; $B = \frac{10}{4} = -\frac{5}{2}$

$C(2+1)(2+2) = 10$; $C(3)(4) = 10$; $C = \frac{10}{12} = \frac{5}{6}$

$\therefore A = \frac{5}{3} \quad B = -\frac{5}{2} \quad C = \frac{5}{6}$

$\therefore I = \frac{5}{3}\int \frac{dx}{x+1} - \frac{5}{2}\int \frac{dx}{x+2} + \frac{5}{6}\int \frac{dx}{x-2}$

$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C.$

(2) Evaluate the following integrals:

(i) $\int \frac{1-x^2}{x(1-2x)}\,dx$ (ii) $\int \frac{x^3+x+1}{x^2-1}\,dx$ (iii) $\int \frac{x^2+8x+4}{x^3-4x}\,dx$

(iv) $\int \frac{2x^3-3x^2-9x+1}{2x^2-x-10}\,dx$ (v) $\int \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)}$

(vi) $\int \frac{bx+C}{(x-p)(x-q)(x-r)}\,dx$; p, q, r are distinct.

(i) $\quad I = \int \frac{1-x^2}{x(1-2x)}\,dx = \int \frac{1-x^2}{x-2x^2}\,dx$

$$-2x^2+x \,\big)\, -x^2+1 \,\big(\, \frac{1}{2}$$
$$\underline{\cancel{-x^2} + \frac{1}{2}x}$$
$$(+) \quad (-)$$
$$0 - \frac{x}{2} + 1$$

$\frac{-x^2}{-2x^2} = \frac{1}{2}$

$$1-x^2 = \frac{1}{2}(x-2x^2) + \left(1-\frac{x}{2}\right) = \frac{1}{2}(x-2x^2) + \left(\frac{2-x}{2}\right)$$

$$\therefore \frac{1-x^2}{x-2x^2} = \frac{\frac{1}{2}(x-2x^2)}{x-2x^2} + \frac{1}{2}\frac{2-x}{(x-2x^2)}$$

$$\therefore I = \frac{1}{2}\int dx + \frac{1}{2}\int \frac{2-x}{x(1-2x)}\,dx = \frac{1}{2}x + \frac{1}{2}I_1$$

$$I_1 = \int \frac{2-x}{x(1-2x)}\,dx \quad \text{let} \quad \frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$$

$$\Rightarrow \quad A(1-2x) + B(x) = 2-x$$

put $x=0$ $\quad A = 2$

put $x = \frac{1}{2}$ $\quad B \times \frac{1}{2} = 2 - \frac{1}{2} \Rightarrow B = 3 \quad \therefore A=2,\ B=3$

$$\therefore I_1 = 2\int \frac{dx}{x} + 3\int \frac{dx}{1-2x} = 2\log|x| + 3\,\frac{\log|1-2x|}{-2}$$

$$= 2\log|x| - \frac{3}{2}\log|1-2x|$$

$$\therefore I = \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + c.$$

(ii) $\quad I = \int \dfrac{x^3+x+1}{x^2-1}\,dx$

$$x^2-1 \,\big)\, \cancel{x^3}+x+1 \,\big(\, x \qquad\qquad \frac{x^3}{x^2} = x$$

$$\underline{\cancel{x^3} \;\; \underset{(+)}{-x}}_{(-)}$$

$$2x+1$$

$\therefore \quad x^3+x+1 = x(x^2-1) + 2x+1$

$\therefore \quad \dfrac{x^3+x+1}{x^2-1} = \dfrac{x\cancel{(x^2-1)}}{\cancel{x^2-1}} + \dfrac{2x+1}{x^2-1} = x + \dfrac{2x+1}{x^2-1}$

$\therefore \quad I = \int x\,dx + \int \dfrac{2x+1}{(x+1)(x-1)}\,dx = \dfrac{x^2}{2} + I_1$

$I_1 = \int \dfrac{2x+1}{(x+1)(x-1)} \qquad$ Let $\dfrac{2x+1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$

$\Rightarrow \quad A(x-1) + B(x+1) = 2x+1$

$x = 1 \qquad B(2) = 3 \qquad \therefore B = \dfrac{3}{2}$

$x = -1 \qquad -2A = -1 \qquad \therefore A = \dfrac{1}{2}$

$I_1 = \dfrac{1}{2}\int \dfrac{dx}{x+1} + \dfrac{3}{2}\int \dfrac{dx}{x-1} = \dfrac{1}{2}\log|x+1| + \dfrac{3}{2}\log|x-1|$

$\therefore \quad I = \dfrac{x^2}{2} + \dfrac{1}{2}\log|x-1| + \dfrac{3}{2}\log|x-1| + c.$

(iii) $\quad I = \int \dfrac{x^2+8x+4}{x^3-4x} = \int \dfrac{x^2+8x+4}{x(x^2-4)} = \int \dfrac{x^2+8x+4}{x(x-2)(x+2)}\,dx$

Let $\dfrac{x^2+8x+4}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$

$$\Rightarrow \quad A(x-2)(x+2) + B(x)(x+2) + C(x)(x-2) = x^2+8x+4$$

$x = 0 \qquad A(-2)(2) = 4 \qquad \therefore A = -1$

$x = -2 \qquad C(-2)(-4) = (-2)^2+8(-2)+4$

$+8C = 4-16+4 = -8$

$C = -1$

$x = 2 \qquad B(2)(4) = 4+16+4$

$8B = 24 \qquad \therefore B = 8$

$\therefore A = -1$
$B = 8$
$C = -1$

$$\therefore I = -\int \frac{dx}{x} + 8\int \frac{dx}{x-2} - \int \frac{dx}{x+2}$$

$$I = -\log|x| + 8\log|x-2| - \log|x+2| + C$$

(iv) $$I = \int \frac{2x^3-3x^2-9x+1}{2x^2-x-10}\,dx$$

$$\begin{array}{r l}
2x^2-x-10 \,\big)\; & \cancel{2x^3} - 3x^2 - 9x + 1 \;\big(x-1 \\
& \underset{(-)}{\cancel{2x^3}} \; \underset{(+)}{-x^2} \; \underset{(+)}{-10x} \\
\hline
& -\cancel{2x^2} + \cancel{x} + 1 \\
& \underset{(+)}{-\cancel{2x^2}} \; \underset{(-)}{\cancel{+x}} \; \underset{(-)}{+10} \\
\hline
& \quad -9
\end{array}$$

$$\frac{2x^3}{2x^2} = x$$

$$\therefore 2x^3-3x^2-9x+1 = (x-1)(2x^2-x-10) - 9$$

$$\therefore \frac{2x^3-3x^2-9x+1}{2x^2-x-10} = \frac{(x-1)\cancel{(2x^2-x-10)}}{\cancel{(2x^2-x-10)}} - \frac{9}{2x^2-x-10}$$

$$= (x-1) - \frac{9}{(2x^2-x-10)}$$

$$\therefore I = \int (x-1)\,dx - 9\int \frac{1}{2x^2-x-10}\,dx$$

$$I = \frac{x^2}{2} - x - 9 I_1$$

$$I_1 = \int \frac{1}{2x^2-x-10}$$

$2x^2-5x+4x-10$ $\qquad$ $10\times 2 = 20$

$x(2x-5)+2(2x-5)$ $\qquad$ 5×4

$(x+2)(2x-5)$

$$= \int \frac{dx}{(x+2)(2x-5)} \qquad \text{Let } \frac{1}{(x+2)(2x-5)} = \frac{A}{(x+2)} + \frac{B}{(2x-5)}$$

$$\Rightarrow \quad A(2x-5) + B(x+2) = 1$$

$x = -2$

$A[-2\times 2-5] = 1 \qquad A = -\frac{1}{9}$

$x = \frac{5}{2} \qquad B\left(\frac{5}{2}+2\right) = 1 \qquad \frac{9B}{2} = 1 \qquad B = \frac{2}{9}$

$$\therefore I_1 = -\frac{1}{9}\int \frac{dx}{x+2} + \frac{2}{9}\int \frac{dx}{2x-5} = -\frac{1}{9}\log|x+2| + \frac{2}{9}\frac{\log|2x-5|}{2}$$

$$\therefore I = \frac{x^2}{2} - x - 9\left[-\frac{1}{9}\log|x+2| + \frac{2}{9}\frac{\log|2x-5|}{2}\right] + C$$

$$= \frac{x^2}{2} - x + \log|x+2| - 2\frac{\log|2x-5|}{2}$$

$$= \frac{x^2}{2} - x + \log|x+2| - \log|2x-5| + C$$

$$= \frac{x^2}{2} - x + \log\left|\frac{x+2}{2x-5}\right| + C.$$

(v) $\quad I = \int \dfrac{ax^2+bx+c}{(x-a)(x-b)(x-c)} \cdot dx.$

$$\frac{ax^2+bx+c}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{x-c}$$

$$A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) = ax^2+bx+c$$

$x=a, \quad x(a-b)(a-c) = a\cdot a^2 + a\cdot a + a = a^3+ab+c$

$\therefore \quad A = \dfrac{a^3+ab+c}{(a-b)(a-c)}$

putting $x=b, c \qquad B = \dfrac{ab^2+b^2+c}{(c-a)(c-b)} \qquad C = \dfrac{ac^2+bc+c}{(c-a)(c-b)}$

$$I = \frac{a^3+ab+c}{(a-b)(a-c)}\int \frac{dx}{x-a} + \frac{ab^2+b^2+c}{(c-a)(c-b)}\int\frac{dx}{x-b} + \frac{ac^2+bc+c}{(c-a)(c-b)}\int\frac{dx}{x-c}$$

$$= \frac{a^3+ab+c}{(a-b)(a-c)}\log|x-a| + \frac{ab^2+b^2+c}{(c-a)(c-b)}\log|x-b| + \frac{ac^2+bc+c}{(c-a)(c-b)}\log|x-c| + C$$

(vi) $\quad I = \int \dfrac{bx+c}{(x-p)(x-q)(x-r)}\, dx$

let $\dfrac{bx+c}{(x-p)(x-q)(x-r)} = \dfrac{A}{(x-p)} + \dfrac{B}{(x-q)} + \dfrac{C}{x-r}$

$$A(x-q)(x-r) + B(x-p)(x-r) + C(x-p)(x-q) = bx+c$$

putting $x = p, q, r$

$A = \dfrac{bp+c}{(p-q)(p-r)} \qquad B = \dfrac{bq+c}{(q-p)(q-r)} \qquad C = \dfrac{br+c}{(r-p)(r-q)}$

$$\therefore \quad I = \frac{bp+c}{(p-q)(p-r)}\log|x-p| + \frac{bq+c}{(q-p)(q-r)}\log|x-q| + \frac{br+c}{(r-p)(r-q)}\log|x-r| + C.$$

3) $I =$ Evaluate $\int \frac{dx}{x(x^n+1)}$

Soln $I = \int \frac{dx}{x(x^n+1)}$ put $x^n = t$ $\quad n x^{n-1} dx = dt$

$dx = \frac{dt}{n x^{n-1}}$

$\therefore I = \frac{1}{n}\int \frac{dt}{x^{n-1} \cdot x(x^n+1)} = \frac{1}{n}\int \frac{dt}{x^n(x^n+1)} = \frac{1}{n}\int \frac{dt}{t(t+1)}$

$= \frac{1}{n}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$

$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + C = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C.$

4) Evaluate : $\int \frac{dx}{x[6(\log x)^2 + 7(\log x) + 2]}$ HINT.. Put $\log x = t$

Ans: $\log\left|\frac{2\log x+1}{3\log x+2}\right| + C.$

5) Evaluate : $\int \frac{\sin x}{(1-\cos x)(2-\cos x)}$ HINT.. put $\cos x = t$

Ans: $\log\left|\frac{\cos x - 1}{\cos x - 2}\right| + C.$

6) Evaluate : $\int \frac{dx}{\sin x + \sin 2x}$ HINT... put $\cos x = t$

Ans : $\frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + C.$

7) Evaluate : $\int \frac{dx}{\sin x(3+2\cos x)}$ HINT...put $\cos x = t$

Ans : $\frac{1}{10}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{2}{5}|3+2\cos x| + C.$

8) Evaluate : $\int \frac{\sin x}{\sin 4x} \cdot dx.$ HINT.. $\sin 4x = 2\sin 2x \cdot \cos 2x$

$\sin x = t$

Ans : $\frac{1}{8}\log|1-\sin x| - \frac{1}{8}\log|1+\sin x| - \frac{1}{4\sqrt{2}}\log|1-\sqrt{2}\sin x| + C.$

Type II : When the denominator contains the repeated linear factors.

1) Evaluate : $\int \frac{x}{(x-1)^2(x+2)}\,dx.$

Sol^n: $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x$

Putting $x=1$, $B(1+2) = 1 \Rightarrow B = \frac{1}{3}$

Putting $x=-2$, $C(-2-1)^2 = -2 \Rightarrow C = -\frac{2}{9}$

Equating coefficient of x^2 $A+C=0$ $A=-C$ $\Rightarrow A = \frac{2}{9}$

$\therefore I = \frac{2}{9}\int \frac{dx}{(x-1)} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$

$= \frac{2}{9}\log|x-1| - \frac{2}{9}\log|x+2| + \frac{1}{3}\frac{(x-1)^{-1}}{-1} + C$

$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C.$

2) Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx.$

Sol^n $I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$

$A(x-1)(x+3) + B(x+3) + C(x-1)^2 = x^2+1$

putting $x=1$, $4B = 2 \Rightarrow B = \frac{1}{2}$

putting $x=-3$ $C(-3-1)^2 = (-3)^2+1$ $16C = 10$ $C = \frac{10}{16} = \frac{5}{8}$

comparing coefficient of x^2, $A+C = 1$ $A = 1-\frac{5}{8} = \frac{3}{8}$

$\therefore A = \frac{3}{8}$ $B = \frac{1}{2}$ $C = \frac{5}{8}$

$$\therefore \frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\frac{1}{(x-1)} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{(x+3)}$$

$$\therefore I = \frac{3}{8}\int\frac{dx}{(x-1)} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{(x+3)}$$

$$= \frac{3}{8}\log|x-1| + \frac{5}{8}\log|x+3| + \frac{1}{2}\frac{(-1)}{x-1} + C$$

$$= \frac{3}{8}\log|x-1| + \frac{5}{8}\log|x+3| - \frac{1}{2(x-1)} + C.$$

3) Evaluate : $\int\frac{3x-2}{(x+3)(x+1)^2}\,.dx$

Ans: $\frac{3x-2}{(x+3)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$A = \frac{-11}{4}$ $B = \frac{11}{4}$ $C = -\frac{5}{2}$

Ans : $\frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C.$

Type III : When the denominator contains non-factorisable quadratic.

1) Evaluate : $\int \frac{2}{(1-x)(1+x^2)} dx$.

let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$A(1+x^2) + (Bx+C)(1-x) = 2$

$A + Ax^2 + Bx - Bx^2 + C - Cx = 2$

$(A-B)x^2 + (B-C)x + A + C = 2$

comparing coefficients $\quad A-B=0 \quad B-C=0 \quad A+C=2$

$A = B \quad B = C$

$\therefore A = B = C = 1.$

$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$

$I = \int \frac{dx}{1-x} + \int \frac{x}{1+x^2} dx + \int \frac{dx}{1+x^2}$

$= \log|1-x| + \frac{1}{2}\int \frac{2x}{1+x^2} dx + \tan^{-1}x + C$

$= \log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C.$

2) Evaluate : $\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$.

Soln $\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \qquad A = \frac{3}{5} \quad B = \frac{2}{5} \quad C = \frac{1}{5}$

ANS : $I = \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C.$

3) Evaluate : $\int \frac{dx}{1+x+x^2+x^3}$

Sol^n: $I = \int \frac{dx}{1+x+x^2+x^3} = \int \frac{dx}{(1+x)+x^2(1+x)} = \int \frac{dx}{(1+x)(1+x^2)}$

$\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{x^2+1}$ $\quad A = \frac{1}{2} \quad B = -\frac{1}{2} \quad C = \frac{1}{2}$

Ans : $I = \frac{1}{2}\log|1+x| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C.$

4) Evaluate : $\int \frac{dx}{x^3+x^2+x}$

Sol^n: $I = \int \frac{dx}{x^3+x^2+x} = \int \frac{dx}{x(x^2+x+1)}$

$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$ $\quad A = 1, \quad B = -1, \quad C = -1$

Ans. $I = \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$

5) Evaluate : $\int \frac{2x}{x^3-1}dx$

$I = \int \frac{2x}{x^3-1}dx = \int \frac{2x}{(x-1)(x^2+x+1)}dx.$

$\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+x+1)}$ $\quad A = \frac{2}{3} \quad B = -\frac{2}{3} \quad C = \frac{2}{3}$

Ans: $I = \frac{2}{3}\log|x-1| - \frac{1}{3}\log|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$

6) Evaluate : $\int \frac{dx}{x^4-1}$ $\quad$ Ans: $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C.$

7) Evaluate : $\int \frac{\tan\phi + \tan^3\phi}{1+\tan^3\phi}d\phi$

Ans: $\frac{1}{6}\log|\tan^2\phi - \tan\phi + 1| - \frac{1}{3}\log|1+\tan\phi| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\phi - 1}{\sqrt{3}}\right) + C.$

Type IV : When only even powers of x occur both in numerator and denominator.

(1) Evaluate: $\displaystyle\int \frac{(x^2+1)}{(x^2+4)(x^2+25)}\,dx$.

Sol^n $\displaystyle I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$

Put $x^2 = y$ so that $\displaystyle\frac{x^2+1}{(x^4+4)(x^2+25)} = \frac{y+1}{(y+4)(y+25)}$

Let $\displaystyle\frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4} + \frac{B}{y+25}$

$A(y+25) + B(y+4) = y+1$

Put $y = -4$, $\quad A(-4+25) = -4+1 \Rightarrow 21A = -3 ; A = -\frac{1}{7}$

$y = -25 \quad B(-25+4) = -25+1 ; \; -21y = -24 ; \; y = \frac{8}{7}$

$\displaystyle\therefore \frac{y+1}{(y+4)(y+25)} = -\frac{1}{(y+4)} + \frac{8}{7(y+25)}$

$\displaystyle\therefore I = -\frac{1}{7}\int \frac{dx}{x^2+(2)^2} + \frac{8}{7}\int \frac{dx}{x^2+(5)^2}$

$\displaystyle = -\frac{1}{7}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{8}{7}\cdot\frac{1}{5}\tan^{-1}\frac{x}{5} + C$

$\displaystyle = -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C.$

(2) Evaluate: $\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$.

Soln

put $x^2 = y$

$$\therefore \frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$$

$$\frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9} \qquad A = -\frac{4}{5} \qquad B = \frac{9}{5}$$

ANS: $I = -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + c$.

3) Evaluate: $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Soln: put $x^2 = t$

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(t+a^2)(t+b^2)} = \frac{A}{t+a^2} + \frac{B}{t+b^2}$$

$$A = \frac{1}{b^2-a^2} \qquad B = \frac{1}{a^2-b^2}$$

Ans: $I = \frac{1}{a^2-b^2}\cdot\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right] + c$.

4) Evaluate: $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$.

Ans: $I = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3\sqrt{2}}\tan^{-1}(\sqrt{2}x) + c$.

5) Evaluate: $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$.

Ans: $x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c$.