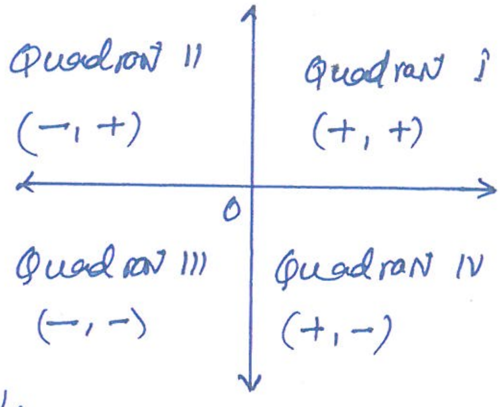
Chapter 10 : Straight Lines (class XI)

Review of the Basic Concepts

1) By using a Cartesian Co-ordinat System, we can specify a point P in the plane with real numbers, also called co-ordinates.

2) The co-ordinates of O, the origin are (0,0).



3) The ordinate of every point on the x-axis is zero.

4) The abscissa of every point on the y-axis is zero.

5) Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

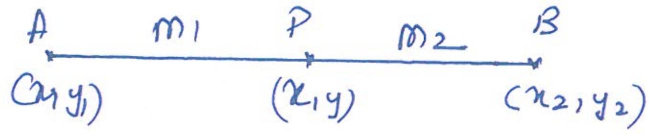
6) The distance between the origin (0,0) and the point (x, y_1) is given by $\sqrt{x^2 + y^2}$.

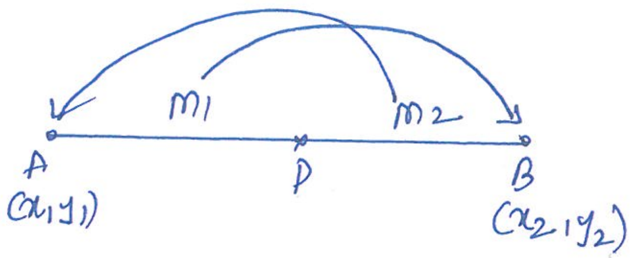
7) To prove that the given three points are collinear prove that the sum of the distance between two points-pair is equal to the distance between the third-point pair.

8) Section formula :

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$



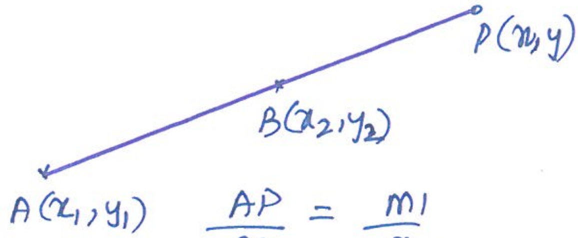


9) Mid-point formula: If P is the mid-point of AB, then $m_1 = m_2$, then the ratio is $1:1$.
$\therefore$ The co-ordinates of the mid-point of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

10) If $P(x, y)$ divides the joints of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $k:1$, then the coordinates are given by

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right)$$

11) <u>External Division</u>:



$$\frac{AP}{PB} = \frac{m_1}{m_2}$$

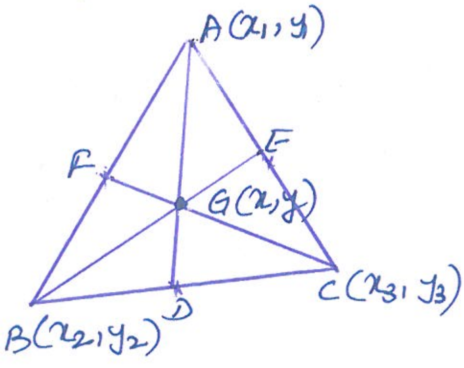
$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$$

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

<u>Note</u>: The result for external division can be obtained from that of internal division by replacing m_2 by $-m_2$.

12) <u>Centroid of a Triangle</u>:
The centroid of a triangle is the intersection of the three medians of the triangle.

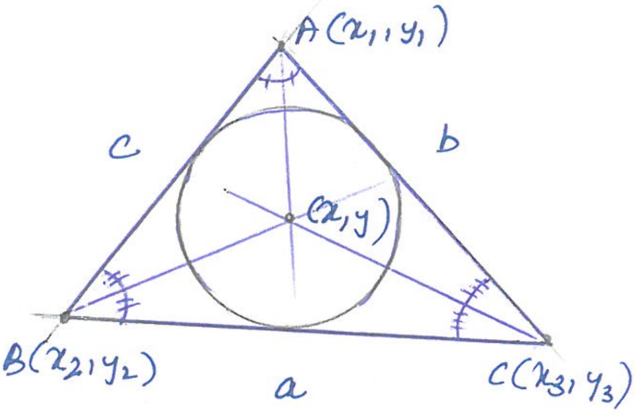
$$x = \frac{x_1 + x_2 + x_3}{3} \qquad y = \frac{y_1 + y_2 + y_3}{3}$$



13) If (a_1, b_1), (a_2, b_2) & (a_3, b_3) are the mid-points of the sides of a triangle, then its centroid is given by $x = \frac{a_1 + a_2 + a_3}{3}$, $y = \frac{b_1 + b_2 + b_3}{3}$

14) Incentre of a Triangle

The point where the three angle bisectors of a triangle meet. The incenter is the center of the triangle's incircle, the largest circle that will fit inside the triangle and touches all three sides.



$$x = \frac{ax_1 + bx_2 + cx_3}{a+b+c}$$

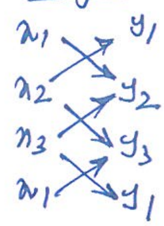
$$y = \frac{ay_1 + by_2 + cy_3}{a+b+c}$$

15) Area of a Triangle

The area of triangle when the co-ordinates of its three vertices are given as (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Rough



16) We take only the numerical value of area.

17) If the area of the triangle ABC is zero, then three points A, B and C lies on a line, i.e. they are collinear.

18) In calculating the area of a polygon by dividing in a number of triangles we take the numerical value of area of each of the triangles.

(19) Locus and its Equations

The path traced by a point moving under a given conditions is called its locus.
A locus is the set of points that satisfy a given geometric conditions.
Every locus will be represented by an equation of the form $f(x,y)=0$, which is called its cartesian equation.

Examples:

(1) A point moves in a plane so as to remain always equidistance from two fixed points $A(-4,-4)$ and $B(2,8)$. Find the equation of its locus. $[x+2y-3=0]$

(2) Find the equation to the locus of point P if the sum of the square of its distances from $(1,2)$ and $(3,4)$ is 25 units.
$[2x^2+2y^2-8x-12y+5=0]$

(3) $A(1,2)$, $B(5,3)$ are two fixed points. Find the locus of a point P, so that $PA:PB=2:3$. $[5x^2+5y^2+22x-12y-91=0$.

(4) Find the equation of the locus of a point which moves so that its distance from the axis of y is away twice its distance from the origin. $[3x^2+4y^2=0]$

(5) a) Find the equation of the locus of all points $P(x,y)$ such that the segment OP has slope 3. $[y=3x]$

(b) Find the locus of a point P which moves so that the sum of the square of its distance from $A(3,0)$ and $B(-3,0)$ is four times the distance between A and B. $[x^2+y^2=3]$

(20) Slope of a Line

1) Slope of a line is not the angle but is the tangent of the angle which the line makes with the positive direction of the x-axis.

2) If a line is parallel to the x-axis, $\theta = 0°$
∴ Its slope $m = \tan 0° = 0$

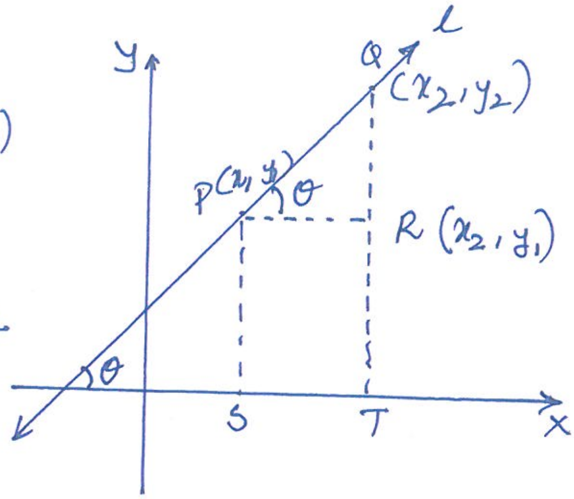
3) If a line is parallel to the y-axis i.e. perpendicular to x-axis, $\theta = 90°$ ∴ Its slope $m = \tan 90° = \infty$ or $\frac{1}{m} = 0$.

4) If a line makes an acute angle with the x-axis, its slope is positive as the tangent of the acute angle is positive.

5) If it makes an obtuse angle, its slope is -ve as the tangent of the obtuse is -ve.

6) Theorem: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line l, then the slope m of the line l is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.

If $x_1 = x_2$ then m is not defined. In that case the line is perpendicular to x-axis.

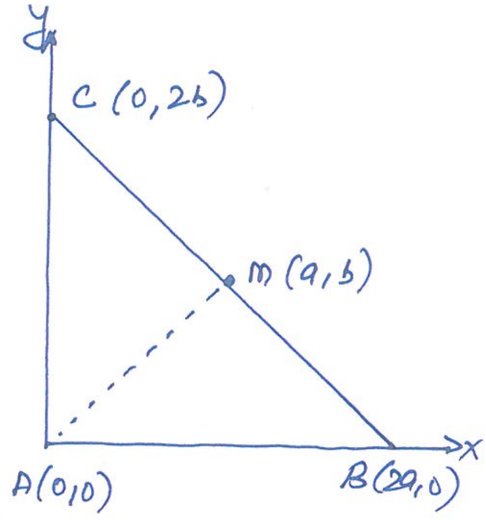


7) Two lines with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$

8) Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. or $m_2 = -\frac{1}{m_1}$

9) Slope $\underline{m} = \tan\theta°$ θ is called inclination

(21) Analytical Proofs of Geometric Theorems



Example 1: Prove analytically that the middle point of the hypotenuse of a right-angle triangle is equidistance from three angular points.

Sol^n: Let co-ordinates of points A(0,0) B(2a,0) and C(0,2b)

$\therefore$ Co-ordinates of point m (the mid-point of B and C is m(a,b).

Using distance formula $mA = \sqrt{a^2+b^2}$

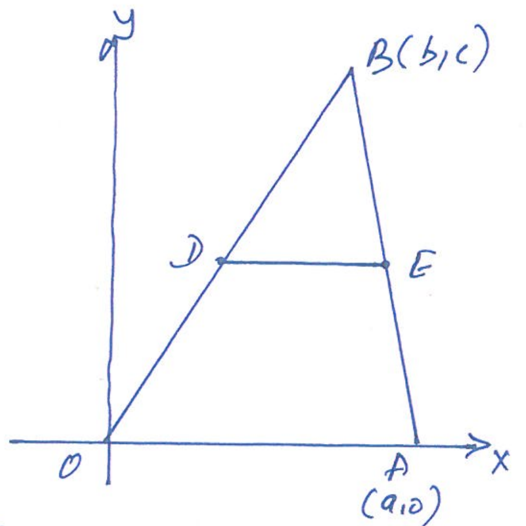
$mB = \sqrt{a^2+b^2}$

$mC = \sqrt{a^2+b^2}$

$\therefore$ $mA = mB = mC$. $\therefore$ m is equidistant from A, B and C.

Example 2

Prove that the line segment joining the mid-point of two sides of a triangle is parallel to the third side and equal to one-half its length.



Sol^n: Consider ΔOAB with vertices O(0,0), A(a,0) & B(b,c).

The mid-point of $OB = \left(\frac{b}{2}, \frac{c}{2}\right)$

The mid-point of E of $AB = \left(\frac{a+b}{2}, \frac{c}{2}\right)$

Slope of $OA\ (m_1) = \dfrac{0-0}{a-0} = \dfrac{0}{a} = 0$

Slope of $DE\ (m_2) = \dfrac{\frac{c}{2} - \frac{c}{2}}{\frac{a+b}{2} - \frac{b}{2}} = 0$

Since $m_1 = m_2$

$\therefore$ $DE \parallel OA$.

i.e. the line joining the mid-point of the two sides is parallel to the third side.

Also $OA = a$

$$DE^2 = \left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2$$

$$DE^2 = \left(\frac{a+\cancel{b}-\cancel{b}}{2}\right)^2 = \sqrt{\frac{a^2}{4}}$$

$$\therefore DE = \frac{a}{2} = \frac{1}{2} OA$$

i.e. the line joining the mid-point of the two sides is half of the third side.

Example 3: Prove that the segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

Example 4: Prove that the diagonals of a parallelogram bisect each other.

Example 5: If the diagonals of a rectangle are perpendicular then prove analytically that it is a square.

Example 6: If the diagonals of a parallelogram are perpendicular, the figure is a rhombus.

Example 7: Prove analytically that the medians to the two equal sides of an isosceles triangles are equal.

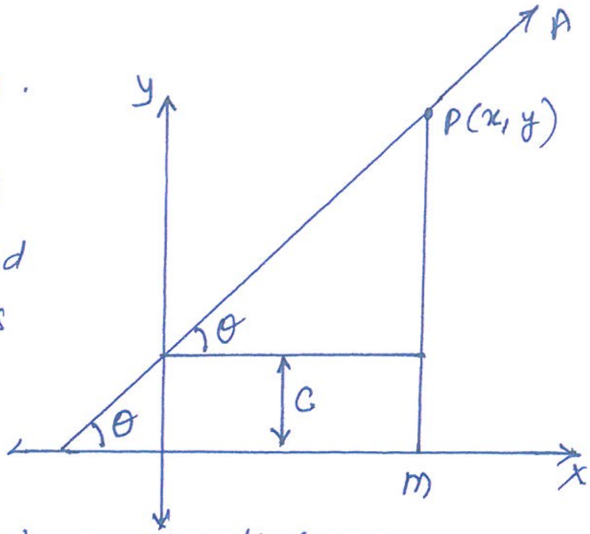
(22) Straight Lines

1) The equation of a straight line parallel to y-axis at a distance h from it is $x = h$.

2) The equation of straight line parallel to the x-axis and at a distance k from it is $y = k$.

3) Equation of the x-axis is $y = 0$

4) Equation of the y-axis is $x = 0$.



5) The equation of line (straight line) having a slope m ($m = \tan\theta$) and making intercept c on the y-axis is $y = mx + c$.

6) $y = mx + c$ is called the slope-intercept form of the equation of a straight line.

7) In case $c = 0$, the line passes through the origin $(0, 0)$, then the equation of the line is $y = mx$.

(23) The Point - Slope form

Article : To find the equation of a straight line through the point (x_1, y_1) and having slope m.

Let the equation of the required line be $y = mx + c$ —(1)

Since the line passes through the point (x_1, y_1)

$$\therefore \quad y_1 = m x_1 + c \qquad \text{—— (2)}$$

Subtracting (2) from (1) and eliminating c.

$$y - y_1 = m(x - x_1)$$

(24) Two Point form

Article : Find the equation of line which passes through two given points (x_1, y_1) and (x_2, y_2)

Let m be the slope of the line. Since it passes through the point (x_1, y_1) its equation is

$$y - y_1 = m(x - x_1) \quad \text{———} (1)$$

Where x_1, y_1 are known but m is unknown.

If (1) passes through (x_2, y_2) then

$$y_2 - y_1 = m(x_2 - x_1)$$

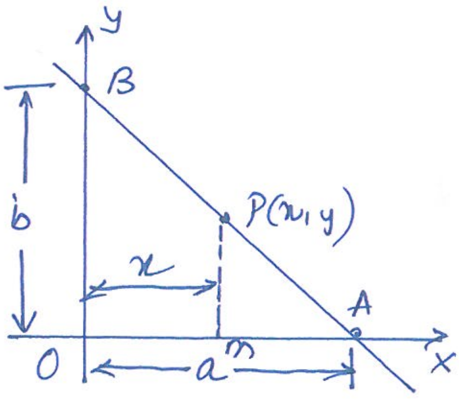
$$\therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting value of m in (1) we get,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(25) Equation of a line in Intercept form

Intercepts of a line



If a line makes intercepts a and b on the co-ordinates axes then the co-ordinates of the points of intersection with the axes are $(a, 0)$ and $(0, b)$.

area of $\Delta OAB = \frac{1}{2}ab$.

Article : To show that the equation of a line which cuts off intercepts 'a' and 'b' on the axes is $\frac{x}{a} + \frac{y}{b} = 1$

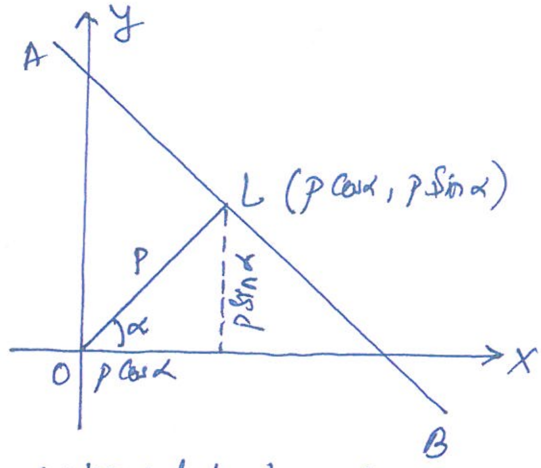
$\Delta AMP \sim \Delta AOB \quad \therefore \quad \frac{y}{b} = 1 - \frac{x}{a} \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

Slope of line $= \frac{-1}{a} \div \frac{1}{b} = -\frac{b}{a}$

class XI

(26) Equation of a line in Normal (Perpendicular) form

The equation of the straight line on which the length of the perpendicular from the origin (0,0) is p and α is the angle which this perpendicular makes with the positive direction of x-axis.



Since $AB \perp OL$ $\therefore$ Slope of AB = -ve reciprocal of slope OL

$\therefore$ Slope of $AB = -\frac{1}{\tan\alpha} = -\cot\alpha$

We know, equation of line passing through (x_1, y_1) and having slope m is given by $y - y_1 = m(x - x_1)$

$\therefore$ Equation of line AB, $y - p\sin\alpha = -\cot\alpha\,(x - p\cos\alpha)$

$$y - p\sin\alpha = -\frac{\cos\alpha}{\sin\alpha}(x - p\cos\alpha)$$

$$y\sin\alpha - p\sin^2\alpha = -x\cos\alpha + p\cos^2\alpha$$

$$x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha)$$

$$\boxed{x\cos\alpha + y\sin\alpha = p}$$ $[\sin^2\alpha + \cos^2\alpha = 1]$

Which is required equation.

Corollary 1: If $\alpha = 0°$ then $x\cos 0° + y\sin 0° = p$, $x(1) + y(0) = p$

$x = p$

(which is the equation of a line parallel to y-axis).

Corollary 2: If $\alpha = \frac{\pi}{2}$, $x\cos\frac{\pi}{2} + y\sin\frac{\pi}{2} = p$, $x(0) + y(1) = p$

$y = p$. which is a line parallel to x-axis.

Corrallery 3: $\alpha = 0$, $p = 0$ $x = 0$, equation of y-axis

Corrallery 4: $\alpha = \frac{\pi}{2}$, $p = 0$ $y = 0$, equation of x-axis.

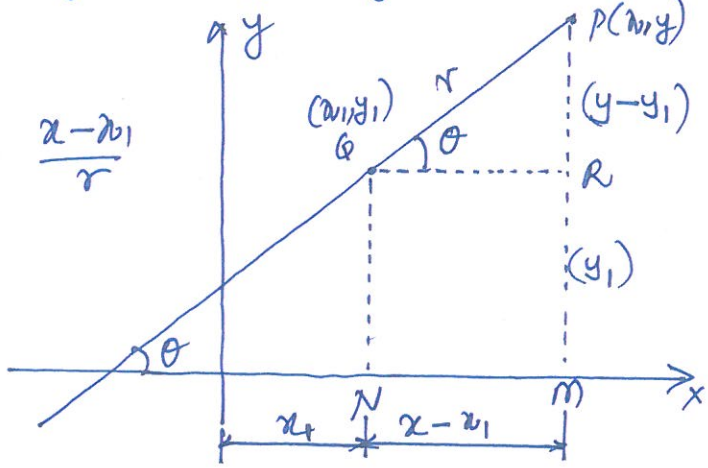
(2.6) Distance or Parametric Form of the Equation of a straight line

The equation of the straight line passing through the point (x_1, y_1) can be expressed in the form

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

where, θ is the angle which the line makes with x-axis
r is the distance of any point (x, y) on the line from (x_1, y_1).

These are called parametric equation of the given line.



$PQ = r \quad \angle PQR = \theta$.

In ΔPQR, $\cos\theta = \frac{QR}{QP} = \frac{NM}{r} = \frac{x-x_1}{r}$

$$r = \frac{x-x_1}{\cos\theta} \quad \text{—— (1)}$$

and $\sin\theta = \frac{RP}{QP} = \frac{y-y_1}{r}$

$$r = \frac{y-y_1}{\sin\theta} \quad \text{—— (2)}$$

From (1) & (2) $\quad \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$

Note: $x = x_1 + r\cos\theta$
$y = y_1 + r\sin\theta$
are called the co-ordinates of any point on this line, in terms of the parameter r.

(27) General Equation of the Straight Line and its Transformation in Different Standard Forms.

Article 1 : Prove that every equation of first degree in x and y always represents a straight line.

The most general equation of first degree in x and y is
$Ax + By + C = 0$. (or) $ax + by + c = 0$.

(28) The general equation $Ax + By + C = 0$ of a straight line is reducible to :

(i) Slope - intercept form : $y = mx + c$

$y = -\frac{A}{B}x - \frac{C}{B}$ Here $m = -\frac{A}{B}$, y-intercept $c = -\frac{C}{B}$

(ii) Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$ $\frac{x}{-C/A} + \frac{y}{-C/B} = 1$

Here the intercept on the axes are $-\frac{C}{A}$ and $-\frac{C}{B}$

(iii) Normal form : $x\cos\alpha + y\sin\alpha = p$ —(1) $Ax + By + C = 0$ —(2)

comparing (1) & (2) $\frac{\cos\alpha}{A} = \frac{\sin\alpha}{B} = \frac{-p}{C}$ [Since line are coincident]

$\cos\alpha = -\frac{Ap}{C}$ $\sin\alpha = -\frac{Bp}{C}$ $\sin^2\alpha + \cos^2\alpha = 1$

$p^2 = \frac{c^2}{a^2+b^2}$ $\Rightarrow$ $p = \frac{c}{\pm\sqrt{a^2+b^2}}$, where p is +ve always.

(a) if c is positive then $p = \frac{c}{\sqrt{a^2+b^2}}$ $\cos\alpha = -\frac{a}{\sqrt{a^2+b^2}}$ $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

The equation reduces to, $-\frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}}$

(b) if c is negative ; $p = -\frac{c}{\sqrt{a^2+b^2}}$

The equation reduces to $\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = -\frac{c}{\sqrt{a^2+b^2}}$

(29) Angle Between two straight lines.

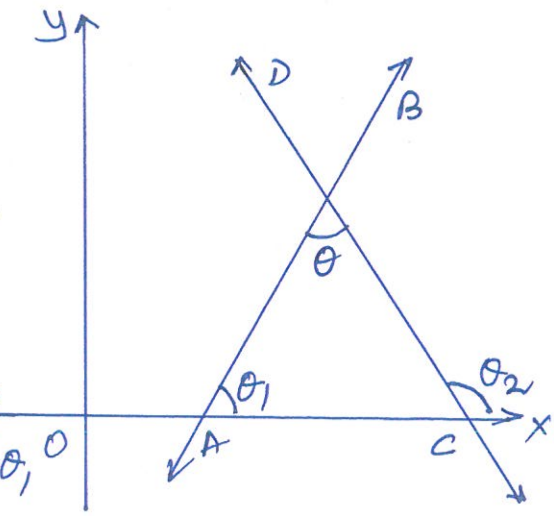
Theorem : The angle θ between the intersecting lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

Let eq of line AB is $y = m_1x + c_1$

eq of line CD ; $y = m_2x + c_2$

$m_1 = \tan\theta_1$ & $m_2 = \tan\theta_2$



Since, Ext angle = Sum of int. opp ∠.

$\therefore$ $\theta_2 = \theta_1 + \theta \Rightarrow \theta = \theta_2 - \theta_1$

$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1}$

$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$

Acute angle $\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$ obtuse angle $\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$.

(i) lines are parallel if $m_1 = m_2$

(ii) lines are perpendicular if $m_1m_2 = -1$

(30) Angle between two lines

$a_1x + b_1y + c_1 = 0$ $\quad\quad$ $\tan\theta = \pm \dfrac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$

$a_2x + b_2y + c_2 = 0$

(i) lines are parallel if $a_1b_2 - a_2b_1 = 0$ or $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

(ii) lines are perpendicular if $a_1a_2 + b_1b_2 = 0$

(iii) If lines are coincident then, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

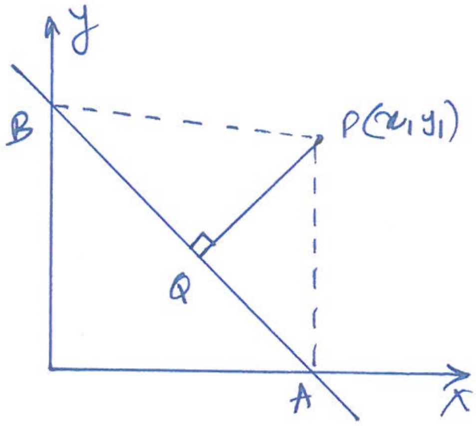
(31) Equation of line parallel to $ax + by + c = 0$
is $ax + by + \lambda = 0$
The constant λ is determined from an additional condition given in the problem.

(32) Equation of line ~~parallel~~ is perpendicular to $ax + by + c = 0$
is $bx - ay + \lambda = 0$

For example, equation of any line $\perp$ to $5x + 3y + 8 = 0$
is $3x - 5y + \lambda = 0$.

(33) Distance of a Point from a Line

Length of perpendicular from the point (x_1, y_1) to the straight line $ax + by + c = 0$ is given by $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$.



AB is given line whose equation is $ax + by + c = 0$

Equation of AB in intercept form $\dfrac{x}{-c/a} + \dfrac{y}{-c/b} = 1$

$\therefore\ A\left(-\dfrac{c}{a}, 0\right) \quad B\left(0, -\dfrac{c}{b}\right)$

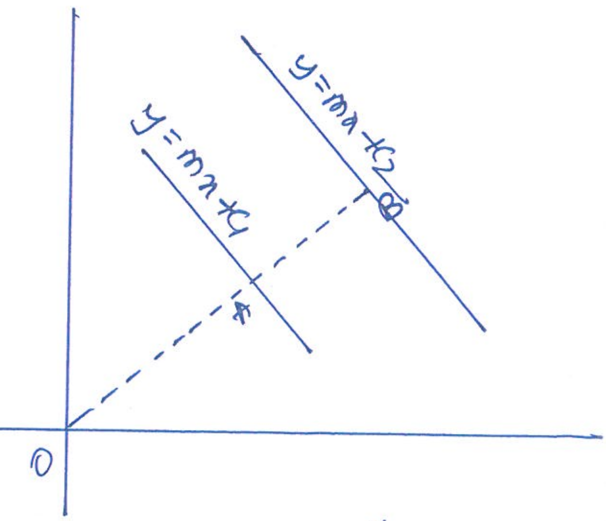
$ar\ \Delta APB = \frac{1}{2} \times AB \times PQ$

$\frac{1}{2} \times AB \times PQ = ar\ \Delta APB$ (Triangle formula - coordinate geometry)

$\therefore\ PQ = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ (Always +ve)

Length of $\perp$ from origin to $ax + by + c = 0$ length of $\perp = \dfrac{c}{\sqrt{a^2+b^2}}$

(34) Distance between parallel lines.



$AB = OB - OA$

$= \dfrac{c_2 - c_1}{\sqrt{1+m^2}}$

To find the distance between two parallel lines of the form $ax + by + c = 0$ and $ax + by + k = 0$, we first find a point on any of these two lines, preferably putting $x = 0$ or $y = 0$ and then we find the $\perp$ distance of this point from the other line.

(35) Point of Intersection of two lines

$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$

Since lines are intersecting $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

i.e. $a_1b_2 - a_2b_1 \neq 0$.

Solving two equations

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

(35) Three lines are said to be concurrent if they pass through a common point i.e. if the point of intersection of any of them lies on the third.

Condition for three lines
$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$
$a_3x + b_3y + c_3 = 0$

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$