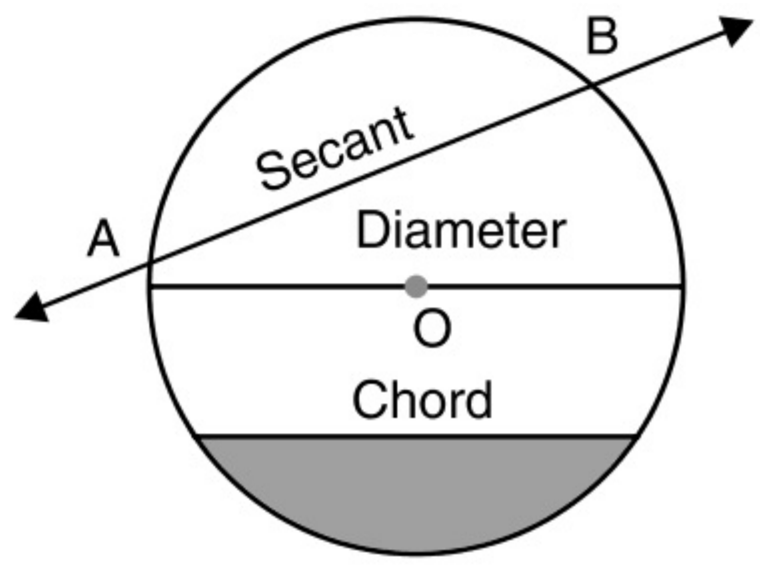
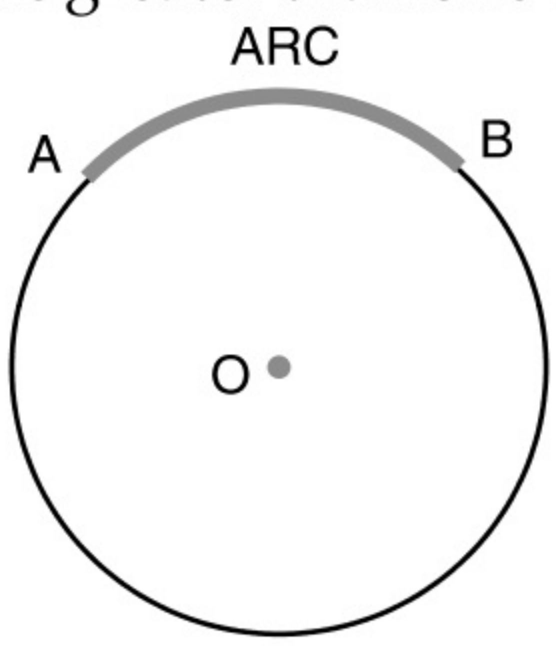
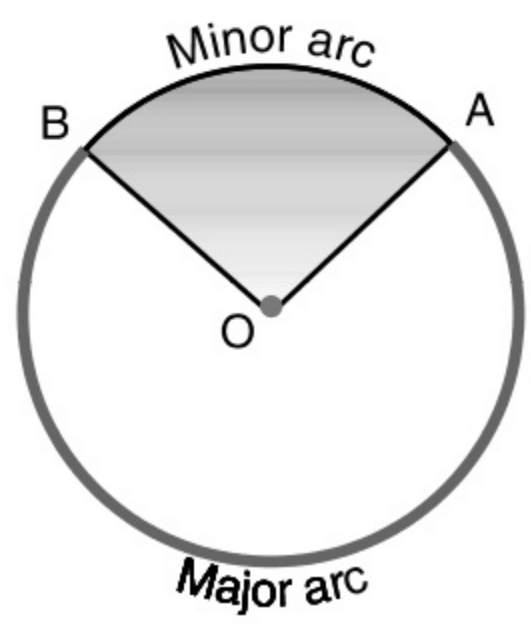
CHAPTER 10 : Circles

- A circle is a collection (set) of all those points in a plane, each one of which is at a constant distance from a fixed point in the plane.
- The fixed point is called the centre and the constant distance is called the radius of the circle.
- A line segment joining two points on a circle is called the chord of the circle.
- A chord passing through the centre of the circle is called a diameter of the circle.
- A line which meets a circle in two points is called a secant of the circle.

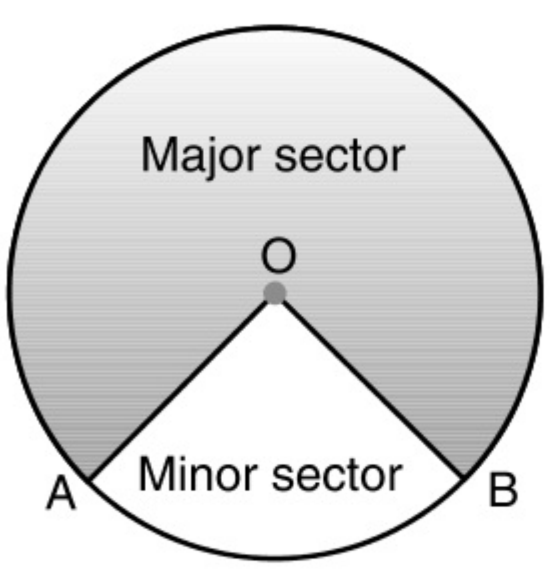


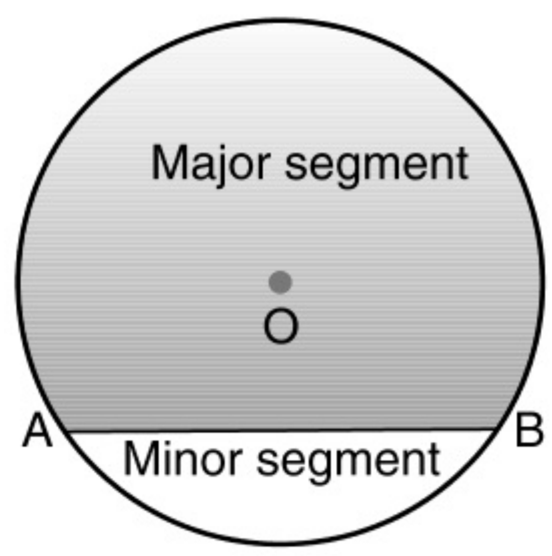
- A (continuous) part of a circle is called an arc of the circle.
- **Minor and Major Arcs :** An arc less than one-half of the whole arc of a circle is called the minor arc of the circle, and an arc greater than one-half of the whole arc of a circle is called the major arc of the circle.



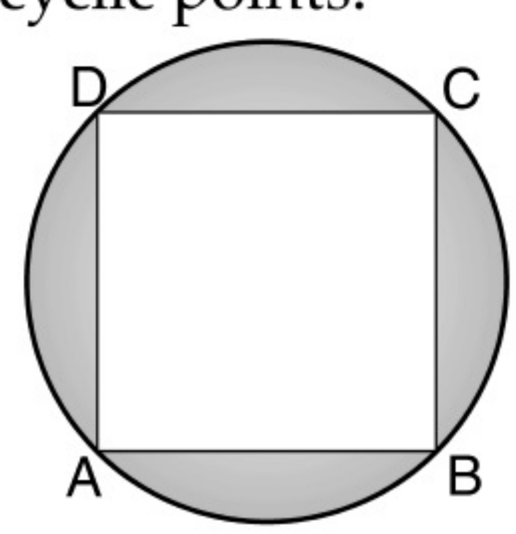


- Diameter = 2 × Radius.
- **Sector of a Circle :** The part of the plane region enclosed by an arc of a circle and its two bounding radii is called a sector of a circle.
- **Segment of a Circle :** A chord of a circle divides it into two parts. Each part is called a segment. The part containing the minor arc is called the minor segment, and the part containing the major arc is called the major segment.





- A quadrilateral of which all the four vertices lie on a circle is called a cyclic quadrilateral. The four vertices *A*, *B*, *C* and *D* are said to be concyclic points.



- The diameter of circle is its longest chord.
- A line can meet a circle at the most two points.
- The degree measure of a semi-circle is 180°.
- The degree measure of a circle is 360°.

Theorems and Proof (wherever required) :

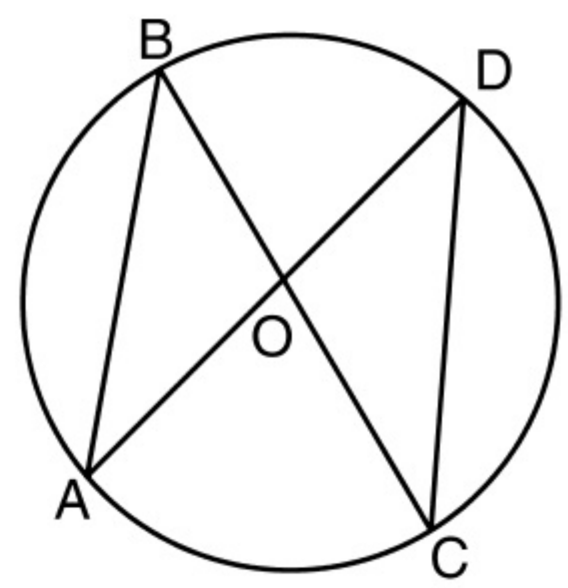
Theorem : 1

Statement : Equal chords of a circle subtend equal angles at the centre.

Given : AB and CD are chords of a circle with centre O, such that AB = CD.

To prove : $\angle AOB = \angle COD$.

Proof : In ΔAOB and ΔCOD,



$AO = CO$ (radii of the same circle)

$BO = DO$ (radii of the same circle)

$AB = CD$ (given)

$\Delta AOB \cong \Delta COD$ (SSS)

Hence, $\angle AOB = \angle COD$ (c.p.c.t.)

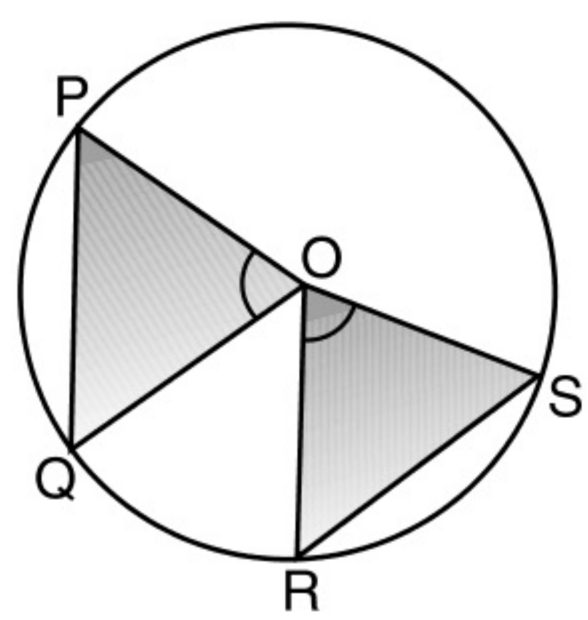
Theorem : 2

Statement : If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.

Given : Two chords PQ and RS of a circle $C(O, r)$, such that $\angle POQ = \angle ROS$.

To prove : $PQ = RS$.

Proof : In ΔPOQ and ΔROS,



$OP = OQ = OR = OS = r$ (radii of the same circle)

and $\angle POQ = \angle ROS$ (given)

$\therefore$ $\Delta POQ \cong \Delta ROS$ (SAS)

$\therefore$ $PQ = RS$. (c.p.c.t.)

Theorem : 3

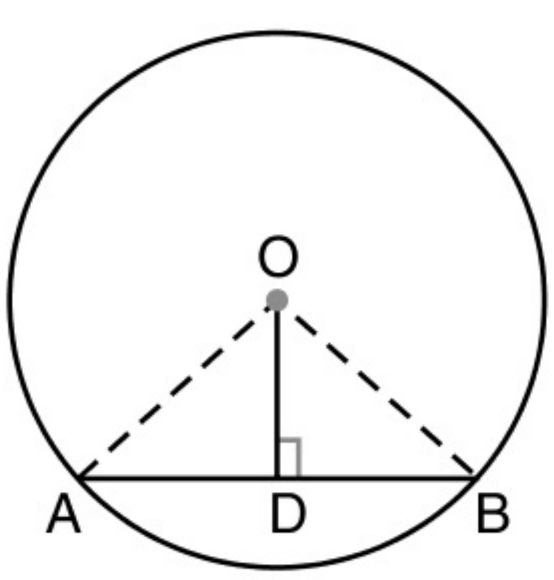
Statement : The perpendicular from the centre of a circle to a chord bisects the chord.

Given : AB is the chord of a circle with centre O and $OD \perp AB$.

To prove : $AD = DB$

Construction : Join OA and OB

Proof : In ΔODA and ΔODB,



$OA = OB$ (radii of the same circle)

$OD = OD$ (common)

$\angle ODA = \angle ODB$ (each is a right angle)

$\Delta ODA \cong \Delta ODB$ (R.H.S.)

$AD = DB$ (c.p.c.t.)

Theorem : 4

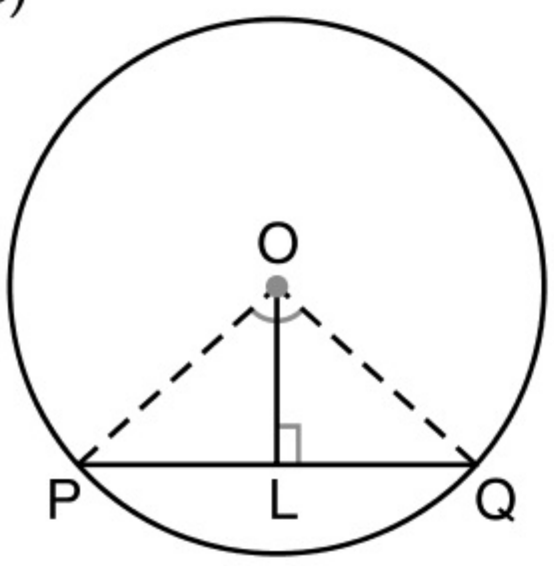
Statement : The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

Given : A chord PQ of a circle $C(O, r)$ and L is the mid-point of PQ.

To prove : $OL \perp PQ$.

Construction : Join OP and OQ.

Proof : In ΔOLP and ΔOLQ,



$OP = OQ$ (radii of the same circle)

$PL = QL$ (given)

$OL = OL$ (common)

$\therefore$ $\Delta OLP \cong \Delta OLQ$ (SSS)

$\therefore$ $\angle OLP = \angle OLQ$ (by c.p.c.t)

Also, $\angle OLP + \angle OLQ = 180°$ (linear pair)

$\therefore$ $\angle OLP = \angle OLQ = 90°$

Hence, $OL \perp PQ$.

Theorem : 5

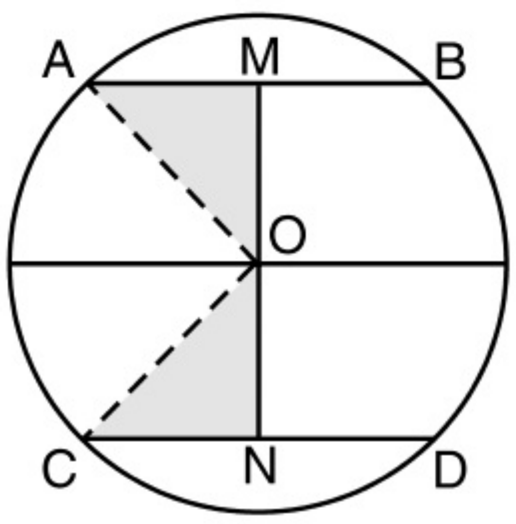
Statement : There is one and only one circle passing through three given non-collinear points.

Theorem : 6

Statement : Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).

Given : AB and CD are two equal chords of a circle.

OM and ON are perpendiculars from the centre to the chords AB and CD.



To prove : $OM = ON$.

Construction : Join OA and OC.

Proof : In ΔAOM and ΔCON,

$OA = OC$ (radii of the same circle)

$MA = CN$ (since OM and ON are perpendicular to the chords and it bisects the two equal chord and $AM = MB$, $CN = ND$)

$\angle OMA = \angle ONC = 90°$

$\therefore$ $\Delta AOM \cong \Delta CON$ (R.H.S.)

$\therefore$ $OM = ON$ (c.p.c.t.)

Equal chords of a circle are equidistant from the centre.

Theorem : 7

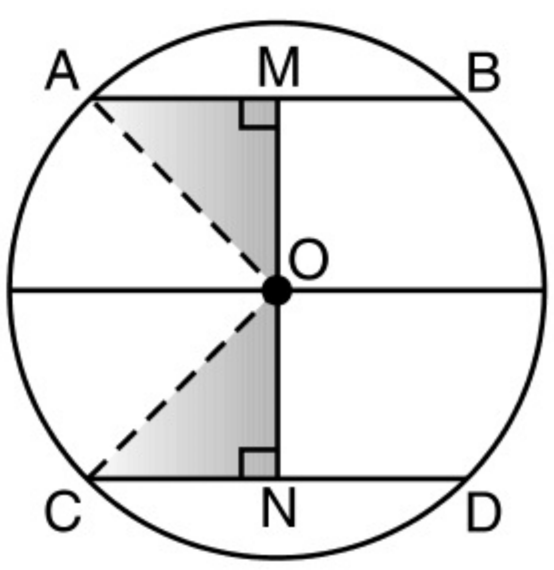
Statement : Chords equidistant from the centre of a circle are equal in length.

Given : OM and ON are perpendiculars from the centre to the chords AB and CD and $OM = ON$.

To prove : Chord AB = Chord CD.

Construction : Join OA and OC.

Proof :



$$OM \perp AB \Rightarrow \frac{1}{2}AB = AM \text{ [Theorem 3]}$$

$$ON \perp CD \Rightarrow \frac{1}{2}CD = CN$$

Consider ΔAOM and ΔCON,

$OA = OC$	(radii of the same circle)
$OM = ON$	(given)
$\angle OMA = \angle ONC = 90°$	(given)
$\Delta AOM \cong \Delta CON$	(RHS congruency)

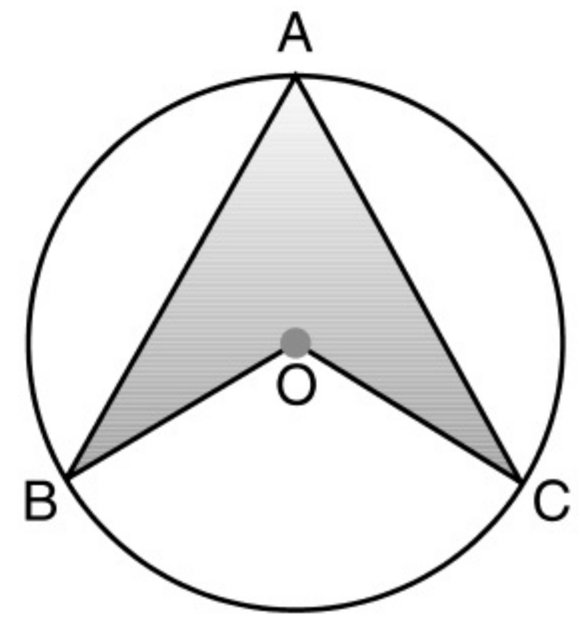
$$AM = CN \Rightarrow \frac{1}{2}AB = \frac{1}{2}CD \Rightarrow AB = CD$$

The two chords are equal if they are equidistant from the centre.

Theorem : 8

Statement : The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Given : O is the centre of the circle.



To prove : $\angle BOC = 2\angle BAC$

Construction : Join O to A.

Construction : Join O to A.

Proof : In ΔAOB, $OA = OB$ (radii of the same circle)

$\Rightarrow$ $\angle 1 = \angle 2$

Similarly, in ΔAOC $\angle 3 = \angle 4$

Now, by exterior angle property, $\angle 5 = \angle 1 + \angle 2$

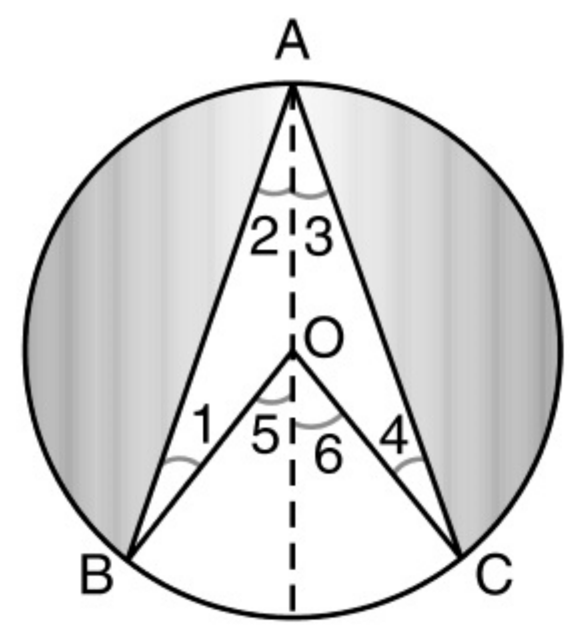
$\angle 6 = \angle 3 + \angle 4$

$\Rightarrow$ $\angle 5 + \angle 6 = \angle 1 + \angle 2 + \angle 3 + \angle 4$

$\Rightarrow$ $\angle 5 + \angle 6 = 2\angle 2 + 2\angle 3$

$= 2(\angle 2 + \angle 3)$

$\Rightarrow$ $\angle BOC = 2\angle BAC$



Theorem : 9

Statement : Angles in the same segment of a circle are equal.

Given : Two angles $\angle ACB$ and $\angle ADB$ are in the same segment of a circle $C(O, r)$.

To prove : $\angle ACB = \angle ADB$

Construction : Join OA and OB.

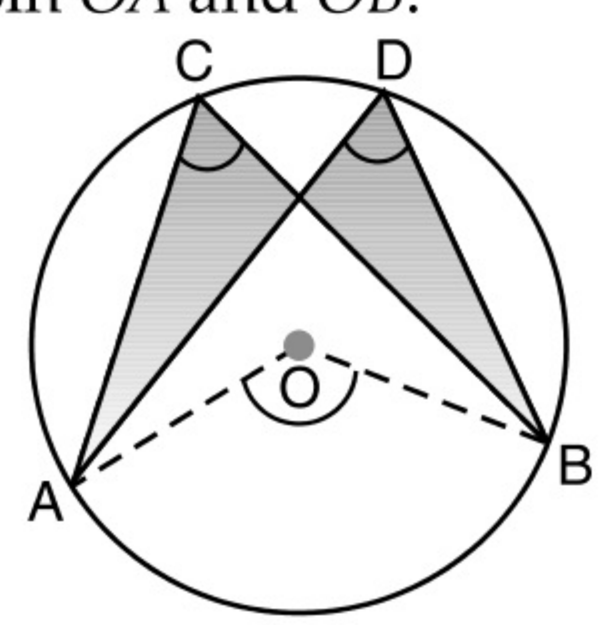


fig. (i)

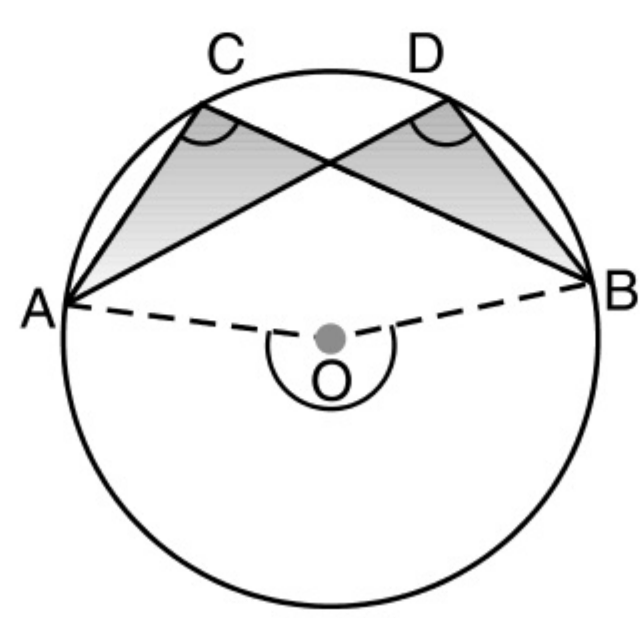


fig. (ii)

Proof : In fig. (i)

We know that, angle subtended by an arc of a circle at the centre is double the angle subtended by the arc in the alternate segment.

Hence, $\angle AOB = 2\angle ACB$

$\angle AOB = 2\angle ADB$

So, $\angle ACB = \angle ADB$

In fig. (ii), we have,

Reflex $\angle AOB = 2\angle ACB$

and Reflex $\angle AOB = 2\angle ADB$

$2\angle ACB = 2\angle ADB$

$\therefore$ $\angle ACB = \angle ADB.$

Theorem : 10

Statement : If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (*i.e.* they are concyclic).

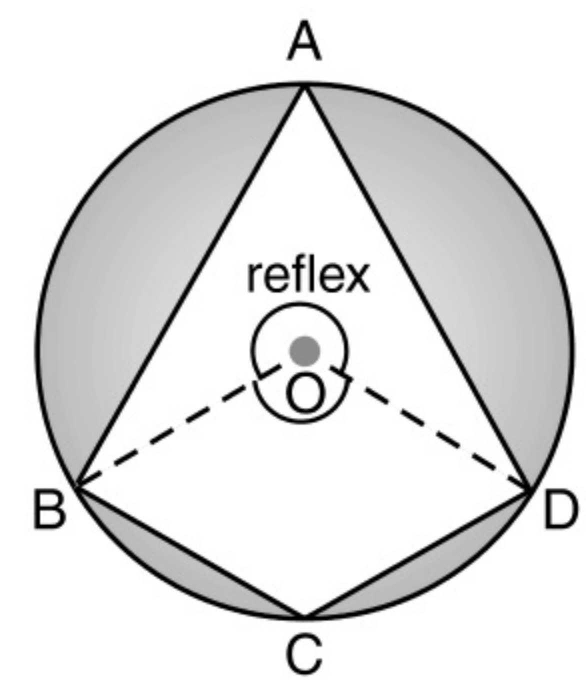
Theorem : 11

Statement : The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

Given : Let $ABCD$ be a cyclic quadrilateral

To prove : $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

Construction : Join OB and OD.



Proof :

$$\angle BOD = 2\,\angle BAD$$

$$\angle BAD = \frac{1}{2}\,\angle BOD$$

Similarly, $$\angle BCD = \frac{1}{2}\text{ reflex }\angle BOD$$

$\therefore$ $$\angle BAD + \angle BCD = \frac{1}{2}\,\angle BOD + \frac{1}{2}\text{ reflex }\angle BOD$$

$$= \frac{1}{2}\,(\angle BOD + \text{reflex }\angle BOD)$$

$$= \frac{1}{2} \times 360°$$

$\therefore$ $\angle A + \angle C = 180°$

Similarly, $\angle B + \angle D = 180°$

Theorem : 12

Statement : If the sum of a pair of opposite angles of a quadrilateral is 180°, then the quadrilateral is cyclic.

■■