CHAPTER 3

TRIGNOMETRY FUNCTIONS

(CLASS XI)

Things - To - Remember (Trigonometric functions - XI)

$1^\circ = 60'$ $1' = 60''$ $\theta(\text{rad}) = \frac{l}{r}$ $\pi(\text{rad}) = 180^\circ$

$\sin(2n\pi + x) = \sin x$ $\cos(2n\pi + x) = \cos x$ $n \in Z$

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

$\cos(-x) = \cos x$ $\sin(-x) = -\sin x$ (a, b) $a = \cos x$ $b = \sin x$

$\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$ $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ $\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$ $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$\cot(A+B) = \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$ $\cot(A-B) = \frac{\cot A \cdot \cot B - 1}{\cot B - \cot A}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$ $\sin 2A = 2 \sin A \cdot \cos B = \frac{2 \tan A}{1 + \tan^2 A}$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ $\sin 3A = 3 \sin A - 4 \sin^3 A$ $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$ $\cos A - \cos B = -2 \sin \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$ $\sin A - \sin B = 2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$ $-2 \sin A \cdot \sin B = \cos(A+B) - \cos(A-B)$

$2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$ $2 \cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

$\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$ $2 \sin^2 A = 1 - \cos 2A$

$\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B)$ $2 \cos^2 A = 1 + \cos 2A$

$\tan(45^\circ + \theta) = \frac{1 + \tan\theta}{1 - \tan\theta}$

$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

Chapter 3 - Trigonometric Functions

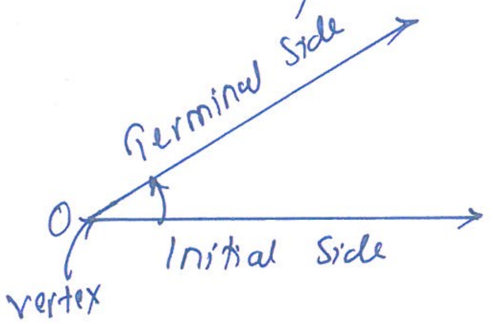
Introduction

Topic - 1

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it mean 'measuring the sides of a triangle.

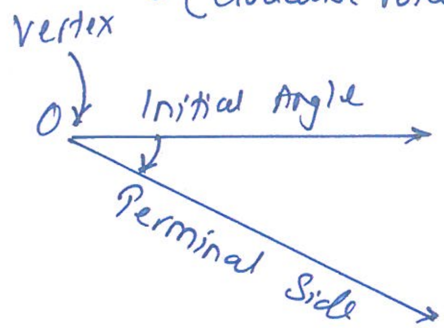
Angles

Angle is a measure of rotation of a given ray about its initial point.



(i) Positive Angle
(anti-clockwise rotation)

(ii) Negative Angle.
(clockwise rotation)

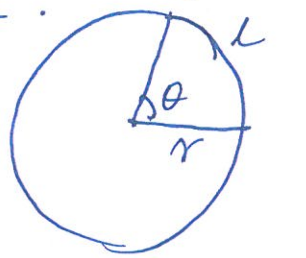


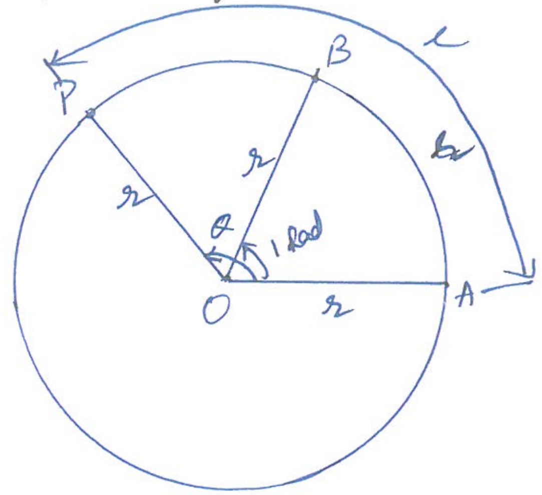
Degree Measure

If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one degree, written as 1°.
A degree is divided into 60 minutes and a minute is divided into 60 seconds.
Thus $1^\circ = 60'$ $\quad$ $1' = 60''$

Radian Measure

Angle subtended at the centre by an arc of length 1 unit in a unit circle (of radius 1 unit) is said to have a measure of 1 radian.
In general, if in a circle of radius r, an arc length l subtends an angle θ radian at the centre.
Thus $\theta = \frac{l}{r}$





To prove $\theta = \frac{l}{r}$

Since angle at centre of circles are proportional.

$$\therefore \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB}$$

$$\frac{\theta}{1 \text{ Rad}} = \frac{l}{r} \qquad \therefore \theta = \frac{l}{r} \text{ radian.}$$

Relation between degree and radian

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is $360°$.

$\therefore$ 2π radian $= 360°$ or π radian $= 180°$

Thus, Radian measure $= \frac{\pi}{180} \times$ Degree measure

(or) Degree measure $= \frac{180}{\pi} \times$ Radian measure.

Note:

(1) When the angle subtended at the centre of a circle is a small one, then: the length of the arc subtending the angle = chord of the arc.

(2) Centesimal system: 1 right angle = 100 grades or 100^g
1 grade or 1^g = 100 min $(100')$
1 min $(1')$ = 100 sec $(100'')$

Exercise - I

(1) Convert $40° 20'$ into radian measure. $\left[\frac{121}{540}\pi\right]$

(2) Convert 6 radian into degree measure. $[343° 38' 11'']$

(3) Find the radius of the circle in which a central angle of $60°$ intercepts an arc length 37.4 cm (use $\pi = \frac{22}{7}$) [35.7 cm]

(4) The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? [$\pi = 3.14$] [6.28 cm]

(5) If the arcs of the same lengths in two circles subtends angles $65°$ and $110°$ at the centre, find the ratio of their radii. [22 : 13]

(6) A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second? [12π radian]

(7) Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (use $\pi = 22/7$). $[12° 36']$

(8) In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord. $\left[\frac{20\pi}{3}\right]$

(9) If in two circles, arcs of the same length subtend angles $60°$ and $75°$ at the centre, find the ratio of their radii. [5 : 4]

(10) Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm.

$\left[\text{(i) } \frac{2}{15} \text{ rad} \quad \text{(ii) } \frac{1}{5} \text{ rad} \quad \text{(iii) } \frac{7}{25} \text{ rad} \right]$

(1) we know that $180^\circ = \pi$ radian

$1^\circ = \frac{\pi}{180}$ rad $\quad 40^\circ 20' = \frac{\pi}{180} \times 40^\circ 20'$

Now $40^\circ 21' = 40^\circ \frac{20}{60} = 40\frac{1}{3} = \frac{121}{3}$

$\therefore 40^\circ 20' = \frac{\pi}{180} \times \frac{121}{3} = \frac{121\pi}{540}$ rad.

(2) we know that π rad $= 180^\circ$

$\therefore 6$ rad $= \frac{180}{\pi} \times 6^\circ = \frac{180}{\cancel{22}\,11} \times 7 \times \cancel{6}\,3$

6 rad $= \frac{180 \times 21}{11} = \frac{3780}{11}$

$= 343\frac{7}{11}$ degree

$= 343^\circ + \frac{7}{11} \times 60'$ $\quad [1^\circ = 60']$

$= 343^\circ + \frac{420'}{11}$

$= 343^\circ + 38' + \frac{2 \times 60''}{11}$ $\quad [1' = 60'']$

$= 343^\circ + 38' + 10.9''$

6 rad $= 343^\circ 38' 10.9''$

$$\begin{array}{r} 343 \\ 11\overline{)3780} \\ 33 \\ \hline 48 \\ 44 \\ \hline 40 \\ 33 \\ \hline 7 \end{array} \qquad \begin{array}{r} 38 \\ 11\overline{)420} \\ 33 \\ \hline 90 \\ 88 \\ \hline 2 \end{array}$$

(3) $l = 37.4$ cm $\quad \theta = 60^\circ = 60^\circ \times \frac{\pi}{180}$ rad $\quad [180^\circ = \pi \text{ rad}]$

$\theta = \frac{\pi}{3}$ rad

we know $\theta = \frac{l}{r}$ $\quad \therefore r = \frac{l}{\theta} = 37.4 \div \frac{\pi}{3}$

$\therefore r = 37.4 \times \frac{3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7$ cm

(4) In 60 min, the minute hand of watch completes one revolution

$\therefore$ 60 min = $360°$

40 min = $\frac{4\cancel{0}}{6\cancel{0}} \times 360° = \frac{2}{3} \times 360° = 2 \times 120 = 240°$

Since $180° = \pi$ rad $\therefore 240° = \frac{\pi}{\cancel{180}_3} \times \cancel{240}^4$ rad.

$240° = \frac{4\pi}{3}$ rad.

$\theta = \frac{l}{r}$ $r = 1.5$ cm $l = \theta \times r$

$l = \frac{4\pi}{\cancel{3}_1} \times \cancel{1.5}^{0.5}$ cm $= 4 \times 3.14 \times 0.5 = 6.28$ cm

$l = 6.28$ cm

(5) $l_1 = l_2 = l$ $\theta_1 = 65°$ $\theta_2 = 110°$

$\theta_1 = \frac{\pi}{180} \times 65°$ $\theta_2 = \frac{\pi}{180} \times 110°$

$l = r_1\theta_1 = r_2\theta_2 \Rightarrow r_1 \times \frac{\cancel{\pi}}{\cancel{180}} \times 65° = r_2 \times \frac{\cancel{\pi}}{\cancel{180}} \times 110°$

$\frac{r_1}{r_2} = \frac{110}{65} = \frac{22}{13}$ $\therefore r_1 : r_2 = 22 : 13$

(6) 1 minute = 360 revolution.

60 sec = 360 Rev

1 sec = $\frac{36\cancel{0}}{6\cancel{0}}$ rev = 6 Rev

1 Rev = $360° = 2\pi$ Rad

$\therefore$ 6 Revolution = $6 \times 2\pi$

= 12π Rad.

(7) $r = 100$ cm $\quad l = 22$ cm

$$\theta = \frac{l}{r} = \frac{22}{100} \text{ rad}$$

π rad $= 180^\circ$

$$\frac{22}{100} \text{ rad} = \frac{180}{\cancel{22}} \times 7 \times \frac{\cancel{22}}{100}$$

$$10\overline{)126}\;\; 12 \qquad \begin{array}{r} 10 \\ \hline 26 \\ 20 \\ \hline 6 \end{array}$$

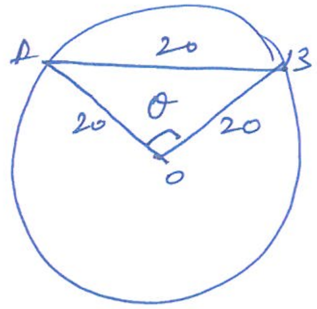
$$= \frac{18 \times 7}{10} = \frac{126}{10}$$

$$= 12^\circ \frac{\cancel{6}^{3}}{\cancel{10}_{5}}$$

$$= 12\frac{3}{5}^\circ = 12^\circ + \frac{3}{5} \times 60'$$

$$= 12^\circ 36'$$

(8) $r = 20$ cm



Since $\triangle AOB$ is equilateral

$$\therefore \theta = 60^\circ = \frac{\pi}{\cancel{180}} \times \cancel{60} \text{ rad}$$

$$\theta = \frac{\pi}{3} \text{ rad} \qquad \theta = \frac{l}{r} \qquad l = \theta \times r$$

$$l = \frac{\pi}{3} \times 20 = \frac{20\pi}{3} \text{ cm}$$

(9) $\theta_1 = 60^\circ = 60 \times \frac{\pi}{180}$ rad $\qquad \theta_2 = 75^\circ = 75 \times \frac{\pi}{180}$ rad

$$l = \theta_1 r_1 = \theta_2 r_2 \qquad 60 \times \frac{\cancel{\pi}}{\cancel{180}} \times r_1 = 75 \times \frac{\cancel{\pi}}{\cancel{180}} \times r_2$$

$$\frac{r_1}{r_2} = \frac{75}{60} = \frac{5}{4} \quad \therefore r_1 : r_2 = 5:4$$

(10) $\theta = \frac{l}{r}$ $\qquad r = 75$ cm

(i) $l = 10$ cm $\qquad \theta_1 = \frac{2}{5}$ rad

(ii) $l = 15$ cm $\qquad \theta_2 = \frac{1}{5}$ rad

(iii) $l = 21$ cm $\qquad \theta_3 = \frac{7}{25}$ rad

Assignment - 1

(1) A wheel rotates making 20 revolution per second. If the radius of the wheel is 35 cm, what linear distance does a point of its rim transverse in three minutes. [7.92 km]

(2) The perimeter of a certain sector of a circle is equal to the length of the arc of the semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds. [$65° 27' 16''$]

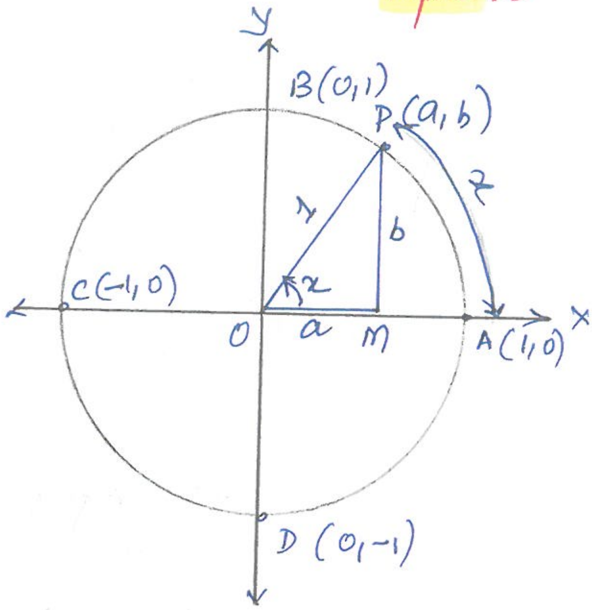
(3) Find the angle between the minute hand and the hour hand of a clock when the time is 7:20. [$100°$]

(4) Karatpur is 64 km from Amritsar. Find to the nearest second the angle subtended at the centre of the earth by the arc joining these two towns, earth being regarded as a sphere of 6400 km radius. [$34' 22''$]

(5) The distance of the moon from the centre of the earth is 385,000 km and moon's diameter subtends an angle of $31'$ at the eye of the observer. Find the diameter of the moon. [3473.1 km nearly]

Topic - 2

Trigonometric Functions



Consider a unit circle with centre at origin of the coordinate axes. Let $P(a,b)$ be any point of circle.

length of arc $\overset{\frown}{AP} = x$

angle $AOP = x$ radian.

$\cos x = a \qquad \sin x = b$

In $\Delta OMP \qquad a^2 + b^2 = 1$

$\therefore \cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre, $\angle AOB = \frac{\pi}{2}$

$\angle AOC = \pi \qquad \angle AOD = \frac{3\pi}{2}$

All angles which are integral multiple of $\frac{\pi}{2}$ are called quadrantal angles.

Since, Points $A(1,0)$, $B(0,1)$, $C(-1,0)$ and $D(0,-1)$

Therefore, $\cos 0^\circ = 1 \qquad \sin 0^\circ = 0 \qquad \cos \pi = -1 \qquad \sin \pi = 0$

$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1 \qquad \cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

If we take one complete revolution from the point P, we again come back to same point P. Thus, if x increases (or decreases) by any integral multiple of 2π, the value of sine and cosine function do not change.

Thus, $\sin(2n\pi + x) = \sin x,\ n \in \mathbb{Z}$, $\cos(2n\pi + x) = \cos x,\ n \in \mathbb{Z}$.

$\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ i.e. x is multiple of π

$\cos x = 0$, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, i.e. x is an odd multiple of $\frac{\pi}{2}$

Thus, $\sin x = 0$ implies $x = n\pi \quad n \in \mathbb{Z}$.

$\cos x = 0$ implies $x = (2n+1)\frac{\pi}{2}$ where n is any integer.

Now, $\operatorname{cosec} x = \dfrac{1}{\sin x}$, $x \neq n\pi \quad n \in \mathbb{Z}$

$\sec x = \dfrac{1}{\cos x}$, $x \neq (2n+1)\frac{\pi}{2}$

$\tan x = \dfrac{\sin x}{\cos x}$, $x \neq (2n+1)\frac{\pi}{2}$

$\cot x = \dfrac{\cos x}{\sin x}$, $x \neq n\pi \quad n \in \mathbb{Z}$.

Also,

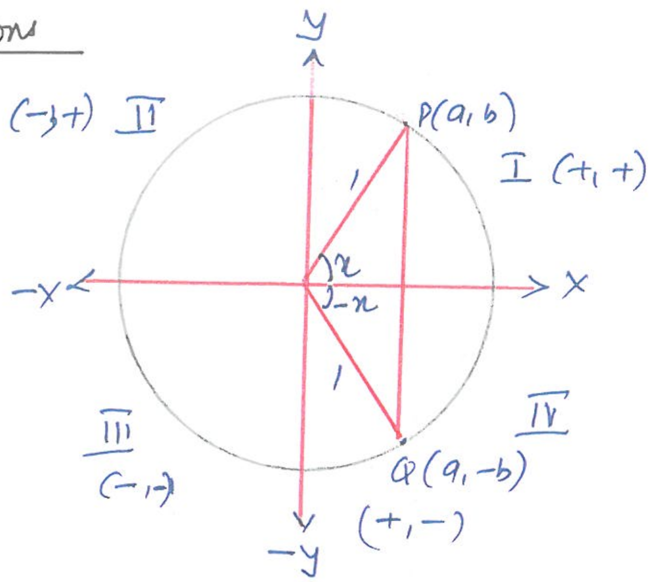
$1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \operatorname{cosec}^2 x$.

	0°	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)	π $(180^\circ$	$\frac{3\pi}{2}$ (270°)	2π (360°)
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not Defined	0	Not Defined	0

Sign of Trigonometric Functions

$\cos(-x) = \cos x$

$\sin(-x) = -\sin x$



Since for every point $p(a, b)$ on the unit circle,

$-1 \leq a \leq 1 \qquad -1 \leq \cos x \leq 1$

$-1 \leq b \leq 1 \qquad -1 \leq \sin x \leq 1$

	I	II	III	IV
sin x	+	+	−	−
cos x	+	−	−	+
tan x	+	−	+	−
cosec x	+	+	−	−
sec x	+	−	−	+
cot x	+	−	+	−

Domain and Range of Trigonometric functions.

Exercise - 2

(1) If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigometric functions.
[$\sec x = -\frac{5}{3}$, $\sin x = -\frac{4}{5}$, $\operatorname{cosec} x = -\frac{5}{4}$, $\tan x = \frac{4}{3}$, $\cot x = \frac{3}{4}$]

(2) If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigometric functions.
[$\tan x = -\frac{12}{5}$, $\sec x = -\frac{13}{5}$, $\cos x = -\frac{5}{13}$, $\sin x = \frac{12}{13}$, $\operatorname{cosec} x = \frac{13}{12}$]

(3) Find the value of $\sin \frac{31\pi}{3}$. [$\frac{\sqrt{3}}{2}$]

(4) Find the value of $\cos(-1710^\circ)$. [0]

(5) Prove that : (i) $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$

(ii) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$.

(6) Prove that : $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta - \tan\theta, & \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\sec\theta + \tan\theta, & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$

(7) Prove that : $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - 1} + \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta + 1} = 2\sec^2\theta$

(8) Prove that : $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

(9) If $\tan A + \sin A = m$ and $\tan A - \sin A = n$ then show that $m^2 - n^2 = \pm 4\sqrt{mn}$.

(10) If $10\sin^4 A + 15\cos^4 A = 6$, find the value of $27\operatorname{cosec}^6 A + 8\sec^6 A$. [250]

(1) Given $\cos x = -\frac{3}{5}$ (Third quadrant) $\quad \sec x = \frac{1}{\cos x} = -\frac{5}{3}$

$\sin^2 x + \cos^2 x = 1 \quad \sin^2 x = 1 - \cos^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

$\sin x = \pm \frac{4}{5}$

$\sin x$ is negative in 3rd quadrant $\therefore \sin x = -\frac{4}{5}$

$\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{5}{4}$

$\tan x = \frac{\sin x}{\cos x} = -\frac{4}{5} \div -\frac{3}{5} = \frac{4}{3}$

$\cot x = \frac{1}{\tan x} = \frac{3}{4}$

(2) Given $\cot x = -\frac{5}{12}$ x lies in 2nd quadrant.

$\tan x = \frac{1}{\cot x} = -\frac{12}{5}$

$\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$, $\sec x = \pm \frac{13}{5}$

$\sec x = -\frac{13}{5}$ ($\sec x$ is $-$ve in 2nd quadrant)

$\cos x = -\frac{5}{13} \quad \sin^2 x = 1 - \cos^2 x = 1 - \frac{25}{169} = \frac{144}{169}$

$\sin x = \pm \frac{12}{13} \quad \sin x = \frac{12}{13}$ ($\sin x$ is $+$ve in 2nd)

$\operatorname{cosec} x = \frac{13}{12}$

(3) We know that value of $\sin x$ repeats after an interval of 2π $\therefore \sin \frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right)$

$= \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

(4) We know that values of $\cos x$ repeats after an interval of 2π or $360°$

$$\cos(-1710°) = \cos(5\times 360 + (-1710°))$$
$$= \cos(1800° - 1710°)$$
$$= \cos 90°$$
$$= 0.$$

(5) (i) $\sec^4 A - \sec^2 A = \sec^2 A(\sec^2 A - 1)$

Since $1 + \tan^2 A = \sec^2 A$ $\therefore = (1+\tan^2 A)(\tan^2 A)$

$\sec^2 A - 1 = \tan^2 A$ $= \tan^2 A + \tan^4 A.$

(ii) $\sin^8\theta - \cos^8\theta = [\sin^4\theta]^2 - [\cos^4\theta]^2$

$$= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$$
$$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)$$
$$= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta)$$

$a^2 + b^2 = (a+b)^2 - 2ab$

$\therefore (\sin^2\theta)^2 + (\cos^2\theta)^2 = [\sin^2\theta + \cos^2\theta]^2 - 2\sin^2\theta\cdot\cos^2\theta$

$= 1 - 2\sin^2\theta\cdot\cos^2\theta$

$\therefore = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

$= LHS.$

(6) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)^2}{1^2-\sin^2\theta}} = \dfrac{1-\sin\theta}{\pm\cos\theta} = \dfrac{1-\sin\theta}{+\cos\theta}$ $\;-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \dfrac{1-\sin\theta}{-\cos\theta}$ $\;\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$-\sec\theta + \tan\theta \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$

(7) $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-1} = \frac{1}{\sin\theta} \div \left(\frac{1}{\sin\theta}-1\right) = \frac{1}{\sin\theta} \div \left(\frac{1-\sin\theta}{\sin\theta}\right)$

$$= \frac{1}{\cancel{\sin\theta}} \times \frac{\cancel{\sin\theta}}{1-\sin\theta} = \frac{1}{1-\sin\theta}$$

$$\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta+1} = \frac{1}{1+\sin\theta}$$

$$\therefore \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-1} + \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta+1} = \frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}$$

$$= \frac{1+\sin\theta}{1^2-\sin^2\theta} + \frac{1-\sin\theta}{1^2-\sin^2\theta} = \frac{1+\cancel{\sin\theta}+1-\cancel{\sin\theta}}{\cos^2\theta}$$

$$= \frac{2}{\cos^2\theta} = 2\sec^2\theta = RHS.$$

(8) $\frac{\tan A+\sec A-1}{\tan A-\sec A+1} = \frac{\frac{\sin A}{\cos A}+\frac{1}{\cos A}-1}{\frac{\sin A}{\cos A}-\frac{1}{\cos A}+1} = \frac{\sin A+1-\cos A}{\sin A-1+\cos A}$

$$= \frac{\sin A-\cos A+1}{\sin A+\cos A-1} = \frac{(\sin A-\cos A+1)(\sin A+\cos A+1)}{(\sin A+\cos A)^2-(1)^2}$$

$$= \frac{\sin^2 A+\cancel{\sin A\cdot\cos A}+\sin A-\cancel{\cos A\sin A}-\cos^2 A-\cancel{\cos A}+\sin A+\cancel{\cos A}+1}{\cancel{\sin^2 A+\cos^2 A}+2\sin A\cdot\cos A\cancel{-1}}$$

$$= \frac{\sin^2 A+\sin A-\cancel{\cos^2 A}+\sin A+\sin^2 A+\cancel{\cos^2 A}}{2\sin A\cdot\cos A}$$

$$= \frac{2\sin^2 A+2\sin A}{2\sin A\cdot\cos A} = \frac{\cancel{2\sin A}(1+\sin A)}{\cancel{2\sin A}\cdot\cos A}$$

$$= \frac{1+\sin A}{\cos A}$$

$$= RHS$$

(9) $\tan A + \sin A = m \qquad \tan A - \sin A = n$

$LHS = m^2 - n^2 = (m+n)(m-n)$

$= (\tan A + \cancel{\sin A} + \tan A - \cancel{\sin A})(\cancel{\tan A} + \sin A - \cancel{\tan A} + \sin A)$

$= 2\tan A \cdot 2 \cdot \sin A$

$= 4 \tan A \cdot \sin A$

$RHS = 4\sqrt{mn} = 4\sqrt{(\tan A + \sin A)(\tan A - \sin A)}$

$= 4\sqrt{\tan^2 A - \sin^2 A} = 4\sqrt{\frac{\sin^2 A}{\cos^2 A} - \sin^2 A}$

$= 4\sqrt{\frac{\sin^2 A - \sin^2 A \cdot \cos^2 A}{\cos^2 A}} = 4\sqrt{\frac{\sin^2 A(1 - \cos^2 A)}{\cos^2 A}}$

$= 4\sqrt{\frac{\sin^2 A \cdot \sin^2 A}{\cos^2 A}} = 4\sqrt{\tan^2 A \cdot \sin^2 A}$

$= \pm 4 \tan A \cdot \sin A$

$= \underline{LHS}$

(10) $10\sin^4 A + 15\cos^4 A = 6$

$10\sin^4 A + 15\cos^4 A = 6(\sin^2 A + \cos^2 A)^2$

Dividing both sides by $\cos^4 A$,

$10\tan^4 A + 15 = 6(\tan^2 A + 1)^2$

$= 6(\tan^4 A + 1 + 2\tan^2 A)$

$4\tan^2 A - 12\tan^2 A + 9 = 0 \qquad (2\tan^2 A - 3)^2 = 0 \qquad \tan^2 A = \frac{3}{2}$

$\cot^2 A = \frac{2}{3} \qquad \sec^2 A = 1 + \tan^2 A = 1 + \frac{3}{2} = \frac{5}{2}$

$\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{2}{3} = \frac{5}{3}$

$27\operatorname{cosec}^6 A + 8\sec^6 A = 27 \times \left(\frac{5}{3}\right)^3 + 8 \times \left(\frac{5}{2}\right)^3$

$= 27 \times \frac{125}{27} + 8 \times \frac{125}{8} = 125 + 125 = 250$

Assigment - 2

(1) Find the values of other five trigometric functions

(a) $\cos x = -\frac{1}{2}$, x lies in third quadrant

(b) $\sin x = \frac{3}{5}$, x lies in second quadrant

(c) $\cot x = \frac{3}{4}$, x lies in third quadrant

(d) $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

(2) Find the values of trignometric function.

(a) $\sin 765^\circ$ (b) $\operatorname{cosec}(-1410^\circ)$ (c) $\tan \frac{19\pi}{3}$ (d) $\sin\left(\frac{-11\pi}{3}\right)$

(3) Prove that : $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$

(4) Check whether $\sin^2 30^\circ$, $\sin^2 45^\circ$, $\sin^2 60^\circ$ are in A.P.

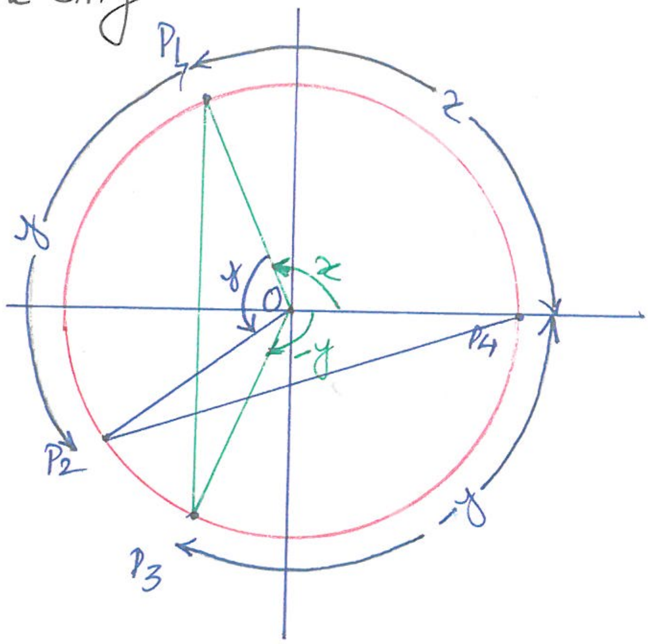
Trigonometric Functions of Sum and Difference of Two Angles

(1) $\sin(-x) = -\sin x$

(2) $\cos(-x) = \cos x$

(3) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin.



$P_4 = (1, 0)$

$P_1 = (\cos x, \sin x)$

$P_2 = [\cos(x+y), \sin(x+y)]$

$P_3 = [\cos(-y), \sin(-y)]$

Since triangles P_1OP_3 and P_2OP_4 are congruent.

$\therefore \quad P_1P_3 = P_2P_4$

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$

Using distance formula,

$$(P_1P_3)^2 = [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$
$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$
$$= \underline{\cos^2 x} + \underline{\cos^2 y} - 2\cos x \cos y + \underline{\sin^2 x} + \underline{\sin^2 y} + 2\sin x \cdot \sin y$$
$$= 2 - 2(\cos x \cos y - \sin x \cdot \sin y) \quad \text{—— (1)}$$

Also,
$$(P_2P_4)^2 = [1 - \cos(x+y)]^2 + [0 - \sin(x+y)]^2$$
$$= 1 + \underline{\cos^2(x+y)} - 2\cos(x+y) + \underline{\sin^2(x+y)}$$
$$= 2 - 2\cos(x+y) \quad \text{—— (2)}$$

Equating (1) & (2)

$$\boxed{\cos(x+y) = \cos x \cos y - \sin x \sin y}$$

(4) $\cos(x-y) = \cos x \cos y + \sin x \cdot \sin y$

Replacing y by $-y$ in identity 3.

$$\cos(x+(-y)) = \cos x \cdot \cos(-y) - \sin x \cdot \sin(-y)$$
$$= \cos x \cos y + \sin x \cdot \sin y \qquad \sin(-x) = -\sin x$$

(5) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ $\qquad \cos\left(\frac{\pi}{2}+x\right) = -\sin x$

(6) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

(7) $\sin(x+y) = \sin x \cos y + \cos x \cdot \sin y$

$$\sin(x+y) = \cos\left[\frac{\pi}{2} - (x+y)\right] = \cos\left[\left(\frac{\pi}{2}-x\right)-y\right]$$
$$= \cos\left(\frac{\pi}{2}-x\right)\cdot\cos y + \sin\left(\frac{\pi}{2}-x\right)\cdot\sin y$$
$$= \sin x \cdot \cos y + \cos x \cdot \sin y$$

(8) $\sin(x-y) = \sin x \cos y - \cos x \cdot \sin y$

(9) $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$ $\qquad \sin\left(\frac{\pi}{2}+x\right) = \cos x$

$\cos(\pi - x) = -\cos x$ $\qquad \sin(\pi - x) = \sin x$

$\cos(\pi + x) = -\cos x$ $\qquad \sin(\pi + x) = -\sin x$

$\cos(2\pi - x) = \cos x$ $\qquad \sin(2\pi - x) = -\sin x$

(10) If none of the angles x, y and $(x+y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the angles is odd multiple of $\frac{\pi}{2}$, it follows that $\cos x$, $\cos y$ and $\cos(x+y)$ are non-zero

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \cdot \sin y}{\cos x \cdot \cos y - \sin x \sin y}$$

Dividing numerator and denominator by $\cos x \cos y$, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \dfrac{\cancel{\cos x} \cdot \sin y}{\cancel{\cos x} \cdot \cos y}}{\dfrac{\cancel{\cos x} \cdot \cancel{\cos y}}{\cancel{\cos x} \cdot \cancel{\cos y}} - \dfrac{\sin x \cdot \sin y}{\cos x \cdot \cos y}}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

(11) $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$$

(12) If none of the angles x, y and $(x+y)$ is multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

(13) $$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

(14) $\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$= \dfrac{1-\tan^2 x}{1+\tan^2 x}$

In $\cos(x+y) = \cos x \cos y - \sin x \cdot \sin y$
put $y = x$

(15) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

In $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

Replace y by x.

(16) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

In $\tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x \cdot \tan y}$
put $y = x$.

(17) $\sin 3x = 3\sin x - 4\sin^3 x$

$\sin 3x = \sin(2x+x)$
...

(18) $\cos 3x = 4\cos^3 x - 3\cos x$

$\cos 3x = \cos(2x+x)$
...

(19) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

$\tan 3x = \tan(2x+x)$

(20) (i) $\cos x + \cos y = 2\cos\frac{x+y}{2} \cdot \cos\frac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\frac{x+y}{2} \cdot \sin\frac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2}$

(21) (i) $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

(ii) $-2\sin x\sin y = \cos(x+y) - \cos(x-y)$

(iii) $2\sin x \cdot \cos y = \sin(x+y) + \sin(x-y)$

(iv) $2\cos x \cdot \sin y = \sin(x+y) - \sin(x-y)$

Exercise - 3

(1) Prove that $3 \sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4 \sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$

(2) Find the value of $\sin 15^\circ$

(3) Find the value of $\tan \frac{13\pi}{12}$

(4) Prove that $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}$

(5) Show that $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

(6) Prove that $\cos\left(\frac{\pi}{4}+x\right) + \cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

(7) Prove that $\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \cot x$

(8) Prove that $\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

(9) Prove that $\sin 70^\circ \cos 10^\circ - \cos 70^\circ \sin 10^\circ = \frac{\sqrt{3}}{2}$

(10) Prove that $\frac{\sin(x+\theta)}{\sin(x+\phi)} = \cos(\theta-\phi) + \cot(x+\phi)\sin(\theta-\phi)$

(11) If $\theta + \phi = 45^\circ$, prove that : $(1+\tan\theta)(1+\tan\phi) = 2$

(12) Prove that $\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$

(13) Prove that : $\tan 3\theta \tan 2\theta \tan\theta = \tan 3\theta - \tan 2\theta - \tan\theta$

(14) Prove that : $\frac{\cos^2 33 - \cos^2 57^\circ}{\sin^2 \frac{21}{2} - \sin^2 \frac{69}{2}} = -\sqrt{2}$

(15) Prove that : $\tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ$

(16) Prove that : $\sin^2 A = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A \cdot \cos B$

(17) If $2\tan\beta + \cot\beta = \tan\alpha$ prove that : $\cot\beta = 2\tan(\alpha-\beta)$

(18) If $\tan A = \frac{1}{\sqrt{x(x^2+x+1)}}$, $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$ $\tan C = \sqrt{x^{-3}+x^{-2}+x^{-1}}$

Prove that $A+B = C$

(19) If $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$ Show that $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$

(20) If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, Prove that $\cos\left(\theta-\frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$

(21) If $\sin\theta = \frac{3}{5}$, $\cos\phi = -\frac{12}{13}$, where θ and ϕ both lie in second quadrant, find the value of $\sin(\theta+\phi)$.

(22) If $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$ (A, B being positive acute), Prove that $A-B = 60°$

(23) Find the maximum and minimum values of $7\cos\theta + 24\sin\theta$.

(24) Prove that $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$ lies between -4 and 10.

Solutions

(1) $3\sin\frac{\pi}{6}\cdot\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\quad \sec\frac{\pi}{3} = 2$

$\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$ $\quad \cot\frac{\pi}{4} = 1$

$$\therefore 3\sin\frac{\pi}{6}\cdot\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cdot\cot\frac{\pi}{4} = 3\times\frac{1}{\cancel{2}}\times\cancel{2} - \overset{2}{\cancel{4}}\times\frac{1}{\cancel{2}}\times 1$$

$$= 3 - 2$$

$$= 1$$

$$= RHS.$$

(2) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

We know $\sin(A-B) = \sin A\cdot\cos B - \cos A\cdot\sin B$

$$\therefore \sin(45^\circ - 30^\circ) = \sin 45\cdot\cos 30 - \cos 45\cdot\sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

(3) $\tan\frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\cdot\tan B}$$

$$\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\times\tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\times\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - 1} = \frac{(\sqrt{3}-1)^2}{2} = \frac{3+1-2\sqrt{3}}{2}$$

$$= \underline{\underline{2-\sqrt{3}}}$$

(4) $LHS = \dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\sin x \cdot \cos y + \cos x \cdot \sin y}{\sin x \cdot \cos y - \cos x \cdot \sin y}$

Dividing numerator and denominator by $\cos x \cdot \cos y$

$$= \dfrac{\dfrac{\sin x \cdot \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}} + \dfrac{\cancel{\cos x} \cdot \sin y}{\cancel{\cos x} \cdot \cos y}}{\dfrac{\sin x \cdot \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}} - \dfrac{\cancel{\cos x} \cdot \sin y}{\cancel{\cos x} \cdot \cos y}} = \dfrac{\tan x + \tan y}{\tan x - \tan x}$$

$= RHS$

(5) $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

$3x = 2x + x$

$\tan 3x = \tan(2x + x)$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$ $\qquad \tan(2x+x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \cdot \tan x}$

$\tan 3x (1 - \tan 2x \cdot \tan x) = \tan 2x + \tan x$

$\tan 3x - \tan 3x \cdot \tan 2x \cdot \tan x = \tan 2x + \tan x$

$\tan 3x \cdot \tan 2x \cdot \tan x = \tan 3x - \tan 2x - \tan x$

(6) $\cos(\frac{\pi}{4} + x) + \cos(\frac{\pi}{4} - x) = \sqrt{2} \cos x$

Using $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

$\cos(\frac{\pi}{4} + x) + \cos(\frac{\pi}{4} - x) = 2 \cdot \cos \left[\dfrac{\frac{\pi}{4} + \cancel{x} + \frac{\pi}{4} - \cancel{x}}{2}\right] \cdot \cos \left[\dfrac{\cancel{\frac{\pi}{4}} + x - \cancel{\frac{\pi}{4}} + x}{2}\right]$

$= 2 \cos \frac{\pi}{4} \times \cos x$

$= 2 \times \frac{1}{\sqrt{2}} \cdot \cos x$

$= \sqrt{2} \cdot \cos x$

$= RHS.$

(7) $\dfrac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \cot x$

Using, $\cos A + \cos B = 2 \cdot \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

$\sin A - \sin B = 2\cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

$$\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \frac{\cancel{2} \cos \cancel{\frac{7x+5x}{2}} \cdot \cos \frac{7x-5x}{2}}{\cancel{2} \cdot \cos \cancel{\frac{7x+5x}{2}} \cdot \sin \frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x$$

(8) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

$\sin 5x + \sin x = 2 \cdot \sin \frac{5x+x}{2} \cdot \cos \frac{5x-x}{2} = 2 \cdot \sin 3x \cdot \cos 2x$

$\cos 5x - \cos x = -2 \cdot \sin \frac{5x+x}{2} \cdot \sin \frac{5x-x}{2} = -2 \sin 3x \cdot \sin 2x$

$$\text{LHS} = \frac{\cancel{2} \sin 3x \cdot \cos 2x - \cancel{2} \sin 3x}{-\cancel{2} \sin 3x \cdot \sin 2x} = -\frac{\cancel{\sin 3x} (\cos 2x - 1)}{\cancel{\sin 3x} \cdot \sin 2x}$$

$$= \frac{1 - \cos 2x}{\sin 2x} = \frac{1 - (\cos^2 x - \sin^2 x)}{2 \cdot \sin x \cdot \cos x}$$

$$= \frac{\sin^2 x + \cancel{\cos^2 x} - \cancel{\cos^2 x} + \sin^2 x}{2 \cdot \sin x \cdot \cos x} = \frac{\cancel{2} \cdot \sin^{\cancel{2}} x}{\cancel{2} \cancel{\sin x} \cdot \cos x}$$

$$= \frac{\sin x}{\cos x}$$

$= \tan x$

$= \text{RHS}.$

(9) $\sin 70^\circ \cdot \cos 10^\circ - \cos 70^\circ \cdot \sin 10^\circ = \frac{\sqrt{3}}{2}$

$$\begin{aligned} LHS &= \sin 70 \cdot \cos 10^\circ - \cos 70^\circ \cdot \sin 10^\circ \\ &= \sin(70^\circ - 10^\circ) \quad [\because \sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B] \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \\ &= RHS. \end{aligned}$$

(10) $\frac{\sin(x+\theta)}{\sin(x+\phi)} = \cos(\theta-\phi) + \cot(x+\phi)\cdot\sin(\theta-\phi)$

$$LHS = \frac{\sin(x+\theta)}{\sin(x+\phi)} = \frac{\sin[(x+\phi)+(\theta-\phi)]}{\sin(x+\phi)}$$

$$= \frac{\sin(x+\phi)\cdot\cos(\theta-\phi) + \cos(x+\phi)\cdot\sin(\theta-\phi)}{\sin(x+\phi)}$$

= [illegible]

$$= \cos(\theta-\phi) + \cot(x+\phi)\cdot\sin(\theta-\phi)$$

$= R.H.S.$

(11) Given $\theta + \phi = 45^\circ$

To prove : $(1+\tan\theta)(1+\tan\phi) = 2$

$$\tan(\theta+\phi) = \tan 45^\circ$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = 1$$

$$\tan\theta + \tan\phi = 1 - \tan\theta\cdot\tan\phi$$

$$1 + \tan\theta + \tan\phi + \tan\theta\cdot\tan\phi = 1+1$$

$$(1+\tan\theta) + \tan\phi(1+\tan\theta) = 2$$

$$(1+\tan\theta)(1+\tan\phi) = 2$$

(12) $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$

$$LHS = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$$

$$= \tan(45^\circ + 11^\circ) \qquad \left[\because \tan(45^\circ + \theta) = \frac{1 + \tan\theta}{1 - \tan\theta}\right]$$

$$= \tan 56^\circ$$

(13) Same as (5)

(14) $\dfrac{\cos^2 33 - \cos^2 57^\circ}{\sin^2 \frac{21}{2} - \sin^2 \frac{69}{2}} = -\sqrt{2}$

$$LHS = \frac{(1 - \sin^2 33^\circ) - (1 - \sin^2 57)}{\sin^2 \frac{21^\circ}{2} - \sin^2 \frac{69^\circ}{2}} = \frac{\sin^2 57 - \sin^2 33^\circ}{\sin^2 \frac{21}{2} - \sin^2 \frac{69}{2}}$$

$$= \frac{(\sin 57 + \sin 33)(\sin 57 - \sin 33)}{\sin\left(\frac{21+69}{2}\right) . \sin\left(\frac{21-69}{2}\right)} \qquad \left[\because \sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)\right]$$

$$= \frac{\sin 90^\circ . \sin 24^\circ}{\sin 45^\circ \cdot \sin(-24^\circ)} = -\frac{\sin 24}{\frac{1}{\sqrt{2}} \cdot \sin 24} = -\sqrt{2} = RHS.$$

(15) $\tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ$

$$\tan 70^\circ = \tan(50 + 20)$$

$$= \frac{\tan 50 + \tan 20}{1 - \tan 50 \cdot \tan 20^\circ}$$

$$\tan 70^\circ - \cancel{\tan 70^\circ} \cdot \tan 50 \, \cancel{\tan 20^\circ} = \tan 50 + \tan 20^\circ$$

$$\tan 70^\circ = 2\tan 50^\circ + \tan 20^\circ \qquad \tan 50 = \cot(90 - 20) = \cot 20^\circ$$

$$= RHS.$$

(16) $\sin^2 A = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cdot\cos A\cdot\cos B$

$RHS = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cdot\cos A\cdot\cos B$

$= \cos^2 B + \cos(A-B)\left[\cos(A-B) - 2\cos A\cdot\cos B\right]$

$= \cos^2 B + \cos(A-B)\left[\cos A\cdot\cos B + \sin A\cdot\sin B - 2\cos A\cdot\cos B\right]$

$= \cos^2 B + \cos(A-B)\left[\sin A\cdot\sin B - \cos A\cdot\cos B\right]$

$= \cos^2 B - \cos(A-B)\cdot\cos(A+B)$ $\quad\because \cos(A+B)\cdot\cos(A-B)$

$= \cos^2 B - (\cos^2 A - \sin^2 B)]$ $\quad = \cos^2 A - \sin^2 B.$

$= \cos^2 B - \cos^2 A + \sin^2 B$

$= 1 - \cos^2 A$

$= \sin^2 A = \underline{LHS}.$

(17) Given : $2\tan\beta + \cot\beta = \tan\alpha$

prove : $\cot\beta = 2\tan(\alpha-\beta)$

$$2\tan(\alpha-\beta) = 2\left(\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right) = 2\,\frac{2\tan\beta+\cot\beta-\tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}$$

$$= 2\,\frac{\tan\beta+\cot\beta}{1+2\tan^2\beta+1} = \cancel{2}\,\frac{\tan\beta+\frac{1}{\tan\beta}}{\cancel{2}+\cancel{2}\tan^2\beta} = \frac{\tan\beta+\frac{1}{\tan\beta}}{1+\tan^2\beta}$$

$$= \frac{\cancel{1+\tan^2\beta}}{\tan\beta\cancel{(1+\tan^2\beta)}} = \cot\beta$$

Hence $\cot\beta = 2\tan(\alpha-\beta)$.

(18) $\tan A = \dfrac{1}{\sqrt{x(x^2+x+1)}}$ $\quad \tan B = \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}$ $\quad \tan C = \sqrt{x^{-3}+x^{-2}+x^{-1}}$

Prove $A + B = C$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\dfrac{1}{\sqrt{x(x^2+x+1)}} + \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \dfrac{1}{\cancel{\sqrt{x}}(x^2+x+1)} \times \dfrac{\cancel{\sqrt{x}}}{\sqrt{x^2+x+1}}}$$

$$\tan(A+B) = \frac{1+x}{\sqrt{x(x^2+x+1)}} \div \frac{\sqrt{\cancel{x}(x^2+x+1)} \times \sqrt{x^2+x+1} - \cancel{\sqrt{x}}}{\sqrt{\cancel{x}(x^2+x+1)} \cdot \sqrt{x^2+x+1}}$$

$$= \frac{1+x}{\sqrt{x(x^2+x+1)}} \times \frac{\sqrt{x^2+x+1} \times \sqrt{x^2+x+1}}{x^2+x+\cancel{1}-\cancel{1}}$$

$$= \frac{\cancel{(1+x)} \times \sqrt{x^2+x+1} \times \cancel{\sqrt{x^2+x+1}}}{\sqrt{x}\,\cancel{\sqrt{x^2+x+1}} \times x\cancel{(x+1)}}$$

$$= \frac{\sqrt{x^2+x+1}}{\sqrt{x} \times x} = \frac{\sqrt{x^2+x+1}}{\sqrt{x} \times \sqrt{x} \times \sqrt{x}}$$

$$= \sqrt{\frac{x^2+x+1}{x \times x \times x}} = \sqrt{x^{-3}(x^2+x+1)}$$

$$= \sqrt{x^{-1}+x^{-2}+x^{-3}}$$

$= \tan C$

$=$ RHS.

$\tan(A+B) = \tan C$

$\therefore A + B = C$

(19) $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$

Show that $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$

$$\tan(\alpha+\theta) = n\tan(\alpha-\theta)$$

$$\frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)} = \frac{n}{1}$$

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (By compenendo and dividendo rule)

$$\therefore \quad \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)} = \frac{n+1}{n-1}$$

$$\frac{\dfrac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)} + \dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}{\dfrac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)} - \dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}} = \frac{n+1}{n-1}$$

$$\frac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\sin(\alpha-\theta)\cos(\alpha+\theta)}{\sin(\alpha+\theta)\cdot\cos(\alpha-\theta)-\sin(\alpha-\theta)\cos(\alpha+\theta)} = \frac{n+1}{n-1}$$

$$\frac{\sin[\alpha+\cancel{\theta}+\alpha-\cancel{\theta}]}{\sin[\cancel{\alpha}+\theta-\cancel{\alpha}+\theta]} = \frac{n+1}{n-1}$$

$$\frac{\sin 2\alpha}{\sin 2\theta} = \frac{n+1}{n-1}$$

$$(n+1)\sin 2\theta = (n-1)\sin 2\alpha$$

(20) $\tan[\pi \cos\theta] = \cot[\pi \sin\theta]$

Prove $\cos\left(\theta - \frac{\pi}{4}\right) = \pm \frac{1}{2\sqrt{2}}$

$$\frac{\sin(\pi\cos\theta)}{\cos(\pi\cos\theta)} = \frac{\cos(\pi\sin\theta)}{\sin(\pi\sin\theta)}$$

$$\sin(\pi\cos\theta)\cdot\sin(\pi\sin\theta) = \cos(\pi\sin\theta)\cdot\cos(\pi\cos\theta)$$

$$\sin(\pi\cos\theta)\cdot\sin(\pi\sin\theta) - \cos(\pi\sin\theta)\cdot\cos(\pi\cos\theta) = 0$$

[Using $\cos(A+B) = \cos A\cdot\cos B - \sin A\cdot\sin B$]

$$\cos[\pi\cos\theta + \pi\sin\theta] = 0$$

$$\pi\cos\theta + \pi\sin\theta = \pm\frac{\pi}{2}$$

$$\cos\theta + \sin\theta = \pm\frac{1}{2}$$

$$\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{4}\cdot\cos\theta + \sin\frac{\pi}{4}\cdot\sin\theta = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$$

(21) $\sin\theta = \frac{3}{5}$ $\cos\phi = \frac{-12}{13}$ θ & ϕ in 2nd quadrants.

$\sin(\theta+\phi) = ?$

$$\sin(\theta+\phi) = \sin\theta\cdot\cos\phi + \cos\theta\cdot\sin\phi$$

$\sin\theta = \frac{3}{5}$ $\cos\theta = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \pm\frac{4}{5} = -\frac{4}{5}$ (θ in IInd quadrant)

$\sin\phi = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13} = \frac{5}{13}$ (ϕ in IInd Quadrant)

$$\therefore \sin(\theta+\phi) = \frac{3}{5}\times\frac{-12}{13} + \frac{-4}{5}\times\frac{5}{13} = \frac{-36-20}{65} = \frac{-56}{65}$$

(22) $\cos A = \frac{1}{7}$ $\cos B = \frac{13}{14}$ A & B +ve and acute

Prove $A - B = 60^\circ$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{1}{49}} = \sqrt{\frac{48}{49}} = \frac{4\sqrt{3}}{7}$$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{169}{196}} = \sqrt{\frac{27}{196}} = \frac{3\sqrt{3}}{14}$$

$$\cos(A-B) = \frac{1}{7} \times \frac{13}{14} + \frac{4\sqrt{3}}{7} \times \frac{3\sqrt{3}}{14}$$

$$= \frac{13+36}{7 \times 14} = \frac{\cancel{49}\,7}{\cancel{7} \times 14} = \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\cos(A-B) = \cos 60^\circ$$

chap 3 (Trigonometric function) XI

Assignments - 3

Prove that:

1. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$

2. $2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

3. $\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$

4. $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$

5. Find the value of:

(i) $\sin 75°$ (ii) $\tan 15°$

6. Prove the following:

$$\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$$

7. $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

8. $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}=\cot^2 x$

9. $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$

10. $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$

11. $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$

12. $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$

13. $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$

14. $\sin 2x+2\sin 4x+\sin 6x=4\cos^2 x\sin 4x$

15. $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$

16. $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

18. $\dfrac{\sin x-\sin y}{\cos x+\cos y}=\tan\dfrac{x-y}{2}$

19. $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$

20. $\dfrac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}=2\sin x$

21. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

22. $\cot x\cot 2x-\cot 2x\cot 3x-\cot 3x\cot x=1$

23. $\tan 4x=\dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$

24. $\cos 4x=1-8\sin^2 x\cos^2 x$

25. $\cos 6x=32\cos^6 x-48\cos^4 x+18\cos^2 x-1$

Question 1:

$$\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$$

Solution 1:

$$\text{L.H.S} = \sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}$$

$$=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-(1)^2$$

$$=\frac{1}{4}+\frac{1}{4}-1=-\frac{1}{2}$$

= R.H.S.

Question 2:

Prove that $2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

Solution 2:

$$\text{L.H.S.} = 2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}$$

$$=2\left(\frac{1}{2}\right)^2+\operatorname{cosec}^2\left(\pi+\frac{\pi}{6}\right)\left(\frac{1}{2}\right)^2$$

$$=2\times\frac{1}{4}+\left(-\operatorname{cosec}\frac{\pi}{6}\right)^2\left(\frac{1}{4}\right)$$

$$=\frac{1}{2}+(-2)^2\left(\frac{1}{4}\right)$$

$$=\frac{1}{2}+\frac{4}{4}=\frac{1}{2}+1=\frac{3}{2}$$

= R.H.S.

Question 3:

Prove that $\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$

Solution 3:

L.H.S. = $\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}$

$$=\left(\sqrt{3}\right)^2+\operatorname{cosec}\left(\pi-\frac{\pi}{6}\right)+3\left(\frac{1}{\sqrt{3}}\right)^2$$

$$=3+\operatorname{cosec}\frac{\pi}{6}+3\times\frac{1}{3}$$

$$=3+2+1=6$$

= R.H.S

Question 4:

Prove that $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$

Solution 4:

L.H.S. $=2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}$

$$=2\left\{\sin\left(\pi-\frac{\pi}{4}\right)\right\}^2+2\left(\frac{1}{\sqrt{2}}\right)^2+2(2)^2$$

$$=2\left\{\sin\frac{\pi}{4}\right\}^2+2\times\frac{1}{2}+8$$

$$=1+1+8$$

= 10

= R.H.S

Question 5:

Find the value of :

(i) $\sin 75^\circ$
(ii) $\tan 15^\circ$

Solution 5:

(i) $\sin 75^\circ = \sin\left(45^\circ + 30^\circ\right)$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$\left[\sin\left(x+y\right) = \sin x \cos y + \cos x \sin y\right]$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\tan 15^\circ = \tan\left(45^\circ - 30^\circ\right)$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \qquad \left[\tan\left(x-y\right) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\right]$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = \frac{3+1-2\sqrt{3}}{\left(\sqrt{3}\right)^2 - (1)^2}$$

$$= \frac{4-2\sqrt{3}}{3-1} = 2 - \sqrt{3}$$

Question 6:

Prove that $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin\left(x+y\right)$

Solution 6:

$$\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right)$$

$$= \frac{1}{2}\left[2\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right)\right] + \frac{1}{2}\left[-2\sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right)\right]$$

$$= \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4} - x\right) + \left(\frac{\pi}{4} - y\right)\right\} + \cos\left\{\left(\frac{\pi}{4} - x\right) - \left(\frac{\pi}{4} - y\right)\right\}\right]$$

$$+ \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4} - x\right) + \left(\frac{\pi}{4} - y\right)\right\} - \cos\left\{\frac{\pi}{4} - x\right\} - \left(\frac{\pi}{4} - y\right)\right]$$

$$\left[\begin{array}{l}\because\ 2\cos A\cos B=\cos(A+B)+\cos(A-B) \\ -2\sin A\sin B=\cos(A+B)-\cos(A-B)\end{array}\right]$$

$$=2\times\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}\right]$$

$$=\cos\left[\frac{\pi}{4}-(x+y)\right]$$

$$=\sin(x+y)$$

= R.H.S.

Question 7:

Prove that $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

Solution 7:

It is known that $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ and $(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$\text{L.H.S.}=\frac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\frac{\left(\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right)}{\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}=\frac{\left(\dfrac{1+\tan x}{1-\tan x}\right)}{\left(\dfrac{1-\tan x}{1+\tan x}\right)}=\left(\frac{1+\tan x}{1-\tan x}\right)^2=\text{R.H.S.}$$

Question 8:

Prove that $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}=\cot^2 x$

Solution 8:

$$\text{L.H.S.}=\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}$$

$$=\frac{[-\cos x][\cos x]}{(\sin x)(-\sin x)}$$

$$=\frac{-\cos^2 x}{-\sin^2 x}$$

$=\cot^2 x$
$=$ R.H.S.

Question 9:

$$\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$$

Solution 9:

L.H.S. $=\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]$

$=\sin x\cos x[\tan x+\cot x]$

$=\sin x\cos x\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)$

$=(\sin x\cos x)\left[\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right]$

$= 1 =$ R.H.S.

Question 10:

Prove that $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$

Solution 10:

L.H.S. $=\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x$

$=\frac{1}{2}[2\sin(n+1)x\sin(n+2)x+2\cos(n+1)x\cos(n+2)x]$

$=\frac{1}{2}\begin{bmatrix}\cos\{(n+1)x-(n+2)x\}-cis\{(n+1)x+(n+2)x\}\\+\cos\{(n+1)x+(n+2)x\}+\cos\{(n+1)x-(n+2)x\}\end{bmatrix}$

$$\begin{bmatrix}\because -2\sin A\sin B=\cos(A+B)-\cos(A-B)\\ 2\cos A\cos B=\cos(A+B)+\cos(A-B)\end{bmatrix}$$

$=\frac{1}{2}\times 2\cos\{(n+1)x-(n+2)x\}$

$=\cos(-x)=\cos x=$ R.H.S.

Question 11:

Prove that $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$

Solution 11:

It is known that $\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)$

$$\therefore \text{L.H.S.}=\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)$$

$$=-2\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)+\left(\left(\frac{3\pi}{4}-x\right)\right)}{2}\right\}.\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)-\left(\frac{3\pi}{4}-x\right)}{2}\right\}$$

$$=-2\sin\left(\frac{3\pi}{4}\right)\sin x$$

$$=-2\sin\left(\pi-\frac{\pi}{4}\right)\sin x$$

$$=-2\sin\frac{\pi}{4}\sin x$$

$$=-2\times\frac{1}{\sqrt{2}}\times\sin x$$

$$=-\sqrt{2}\sin x$$

= R.H.S.

Question 12:

Prove that $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$

Solution 12:

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),\ \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.}=\sin^2 6x-\sin^2 4x$$

$$=(\sin 6x+\sin 4x)(\sin 6x-\sin 4x)$$

$$=\left[2\sin\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right)\right]\left[2\cos\left(\frac{6x+4x}{2}\right).\sin\left(\frac{6x-4x}{2}\right)\right]$$

$$=(2\sin 5x\cos x)(2\cos 5x\sin x)=(2\sin 5x\cos 5x)(2\sin x\cos x)$$

$$=\sin 10x\sin 2x$$

= R.H.S.

Question 13:

Prove that $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$

Solution 13:

It is known that

$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, $\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\therefore$ L.H.S. $=\cos^2 2x-\cos^2 6x$

$=(\cos 2x+\cos 6x)(\cos 2x-6x)$

$=\left[2\cos\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)\right]\left[-2\sin\left(\frac{2x+6x}{2}\right)\sin\left(\frac{2x-6x}{2}\right)\right]$

$=[2\cos 4x\cos(-2x)][-2\sin 4x\sin(-2x)]$

$=[2\cos 4x\cos 2x][-2\sin 4x(-\sin 2x)]$

$=(2\sin 4x\cos 4x)(2\sin 2x\cos 2x)$

$=\sin 8x\sin 4x=$ R.H.S

Question 14:

Prove that $\sin 2x+2\sin 4x+\sin 6x=4\cos^2 x\sin 4x$

Solution 14:

L.H.S. $=\sin 2x+2\sin 4x+\sin 6x$

$=[\sin 2x+\sin 6x]+2\sin 4x$

$=\left[2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)\right]+2\sin 4x$

$\left[\because\ \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$

$=2\sin 4x\cos(-2x)+2\sin 4x$

$=2\sin 4x\cos 2x+2\sin 4x$

$=2\sin 4x(\cos 2x+1)$

$=2\sin 4x(2\cos^2 x-1+1)$

$=2\sin 4x(2\cos^2 x)$

$=4\cos^2 x\sin\ 4x$

$=$ R.H.S.

Question 15:

Prove that $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$

Solution 15:

L.H.S $=\cot 4x(\sin 5x+\sin 3x)$

$=\frac{\cot 4x}{\sin 4x}\left[2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)\right]$

$$\left[\because \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$=\left(\frac{\cos 4x}{\sin 4x}\right)[2\sin 4x\cos x]$$

$$=2\cos 4x\cos x$$

$$\text{R.H.S.}=\cot x(\sin 5x-\sin 3x)$$

$$=\frac{\cos x}{\sin x}\left[2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)\right]$$

$$\left[\because \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$=\frac{\cos x}{\sin x}[2\cos 4x\sin x]$$

$$=2\cos 4x.\cos x$$

L.H.S. = R.H.S.

Question 16:

Prove that $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$

Solution 16:

It is known that

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),\ \sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.}=\frac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}$$

$$=\frac{-2\sin\left(\frac{9x+5x}{2}\right).\sin\left(\frac{9x-5x}{2}\right)}{2\cos\left(\frac{17x+3x}{2}\right).\sin\left(\frac{17x-3x}{2}\right)}$$

$$=\frac{-2\sin 7x.\sin 2x}{2\cos 10x.\sin 7x}$$

$$=-\frac{\sin 2x}{\cos 10x}$$

$$=\text{R.H.S.}$$

Question 17:

Prove that: $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

Solution 17:

It is known that

$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$,

$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\therefore \text{L.H.S.}=\frac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}$

$=\frac{2\sin\left(\frac{5x+3x}{2}\right).\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right).\cos\left(\frac{5x-3x}{2}\right)}$

$=\frac{2\sin 4x.\cos x}{2\cos 4x.\cos x}$

$=\tan 4x=\text{R.H.S.}$

Question 18:

Prove that $\frac{\sin x-\sin y}{\cos x+\cos y}=\tan\frac{x-y}{2}$

Solution 18:

It is known that

$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$,

$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\therefore \text{L.H.S.}=\frac{\sin x-\sin y}{\cos x+\cos y}$

$=\frac{2\cos\left(\frac{x+y}{2}\right).\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)}$

$=\frac{\sin\left(\frac{x-y}{2}\right)}{\cos\left(\frac{x-y}{2}\right)}$

$=\tan\left(\frac{x-y}{2}\right)=\text{R.H.S.}$

Question 19:

Prove that $\frac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$

Solution 19:

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.}=\frac{\sin x+\sin 3x}{\cos x+\cos 3x}$$

$$=\frac{2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$=\frac{\sin 2x}{\cos 2x}$$

$$=\tan 2x$$

$$=\text{R.H.S.}$$

Question 20:

Prove that $\dfrac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}=2\sin x$

Solution 20:

It is known that

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),\cos^2 A-\sin^2 A=\cos 2A$$

$$\therefore \text{L.H.S.}=\frac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}$$

$$=\frac{2\cos\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}{-\cos 2x}$$

$$=\frac{2\cos 2x\sin(-x)}{-\cos 2x}$$

$$=-2\times(-\sin x)$$

$$=2\sin x=\text{R.H.S.}$$

Question 21:

Prove that $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

Solution 21:

$$\text{L.H.S.}=\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}$$

$=\dfrac{(\cos 4x+\cos 2x)+\cos 3x}{(\sin 4x+\sin 2x)+\sin 3x}$

$=\dfrac{2\cos\left(\dfrac{4x+2x}{2}\right)\cos\left(\dfrac{4x-2x}{2}\right)+\cos 3x}{2\sin\left(\dfrac{4x+2x}{2}\right)\cos\left(\dfrac{4x-2x}{2}\right)+\sin 3x}$

$\left[\because \cos A+\cos B=2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right), \sin A+\sin B=2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)\right]$

$=\dfrac{2\cos 3x\cos +\cos 3x}{2\sin 3x\cos x+\sin 3x}$

$=\dfrac{\cos 3x(2\cos x+1)}{\sin 3x(2\cos x+1)}$

$\cot 3x=\text{R.H.S.}$

Question 22:

Prove that $\cot x\cot 2x-\cot 2x\cot 3x-\cot 3x\cot x=1$

Solution 22:

L.H.S. $=\cot x\cot 2x-\cot 2x\cot 3x-\cot 3x\cot x$

$=\cot x\cot 2x-\cot 3x(\cot 2x+\cot x)$

$=\cot x\cot 2x-\cot(2x+x)(\cot 2x+\cot x)$

$=\cot x\cot 2x-\left[\dfrac{\cot 2x\cot x-1}{\cot x+\cot 2x}\right](\cot 2x+\cot x)$

$\left[\because \cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}\right]$

$=\cot x\cot 2x-(\cot 2x\cot x-1)=1=\text{R.H.S.}$

Question 23:

Prove that $\tan 4x=\dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$

Solution 23:

It is known that $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

$\therefore$ L.H.S. $=\tan 4x=\tan 2(2x)$

$=\dfrac{2\tan 2x}{1-\tan^2(2x)}$

$$=\frac{2\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}$$

$$=\frac{\left(\frac{4\tan x}{1-\tan^2 x}\right)}{\left[1-\frac{4\tan^2 x}{\left(1-\tan^2 x\right)^2}\right]}$$

$$=\frac{\left(\frac{4\tan x}{1-\tan^2 x}\right)}{\left[\frac{\left(1-\tan^2 x\right)^2-4\tan^2 x}{\left(1-\tan^2 x\right)^2}\right]}$$

$$=\frac{4\tan x\left(1-\tan^2 x\right)}{\left(1-\tan^2 x\right)^2-4\tan^2 x}$$

$$=\frac{4\tan x\left(1-\tan^2 x\right)}{1+\tan^4 x-2\tan^2 x-4\tan^2 x}$$

$$=\frac{4\tan x\left(1-\tan^2 x\right)}{1-6\tan^2 x+\tan^4 x}=\text{R.H.S.}$$

Question 24:

Prove that: $\cos 4x = 1-8\sin^2 x\cos^2 x$

Solution 24:

L.H.S. $=\cos 4x$

$=\cos 2(2x)$

$=1-2\sin^2 2x\left[\cos 2A = 1-2\sin^2 A\right]$

$=1-2(2\sin x\cos x)^2\left[\sin 2A = 2\sin A\cos A\right]$

$=1-8\sin^2 x\cos^2 x$

$=\text{R.H.S.}$

Question 25:

Prove that: $\cos 6x = 32x\cos^6 x-48\cos^4 x+18\cos^2 x-1$

Solution 25:

L.H.S. $=\cos 6x$

$=\cos 3(2x)$

Trigonometric Equations

Equation involving trigonometric function of a variable are called trigonometric equation.
We know that value of $\sin x$ and $\cos x$ repeat after an interval of 2π and value of $\tan x$ repeat after an interval of π.
The solutions of a trigonometric equation for which $0 \le x \le 2\pi$ are called principal solution.
The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the general solution.

Example-1: Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$

We know $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \sin\frac{2\pi}{3} = \sin(\pi - \frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Principal solutions of $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

Example-2: Find the principal solution of equation $\tan x = -\frac{1}{\sqrt{3}}$

We know $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad \tan(\pi - \frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

$\tan(2\pi - \frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ $\quad$ Thus $\tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

$\therefore$ Principal solution are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$

Theorem 1 : For any real number x and y, $\sin x = \sin y$ implies $x = n\pi + (-1)^n y$, $n \in Z$.

Proof: If $\sin x = \sin y$, then $\sin x - \sin y = 0$

$$\sin x - \sin y = 2\cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2} = 0$$

which give $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

We already seen that, $\sin x = 0$ give $x = n\pi$ $n \in Z$
$\cos x = 0$ give $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

$\therefore \frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$ $n \in Z$.

$x = (2n+1)\pi - y$ $\qquad x = 2n\pi + y$ $n \in Z$.

Hence $x = (2n+1)\pi + (-1)^{2n+1} \cdot y$ $\qquad x = 2n\pi + (-1)^{2n} \cdot y$

Combining these two results, we get
$x = n\pi + (-1)^n \cdot y$ when $n \in Z$.

Theorem 2 : For any real numbers x and y
$\cos x = \cos y$, implies $x = 2n\pi \pm y$, $n \in Z$.

Proof:
$\cos x = \cos y$, $\cos x - \cos y = 0$, $-2\sin\frac{x+y}{2} \cdot \sin\frac{x-y}{2} = 0$

$\sin\frac{x+y}{2} = 0$ (or) $\sin\frac{x-y}{2} = 0$, $\frac{x+y}{2} = n\pi$ $\qquad \frac{x-y}{2} = m\pi$

$x = 2n\pi - y$ or $x = 2n\pi + y$

Hence $x = 2n\pi \pm y$, $n \in Z$.

Theorem 3 : Prove that if x and y are not odd multiple of $\frac{\pi}{2}$, then $\tan x = \tan y$ implies $x = n\pi + y$, $n \in Z$.

Proof: If $\tan x = \tan y$ then $\tan x - \tan y = 0$

$$\frac{\sin x \cdot \cos y - \cos x \cdot \sin y}{\cos x \cdot \cos y} = 0 \qquad \frac{\sin(x-y)}{\cos x \cdot \cos y} = 0.$$

$\cos x \cdot \cos y$ can not be zero, $\therefore \sin(x-y) = 0$

$\therefore x - y = n\pi$

$x = n\pi + y$

Example 3: Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$

Sol^n: $-\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

$\therefore \sin x = \sin\frac{4\pi}{3}$ $\quad \therefore x = n\pi + (-1)^n\frac{4\pi}{3}$ $\quad n \in Z.$

(General Solution).

Example 4: Solve $\cos x = \frac{1}{2}$

Sol^n: $\frac{1}{2} = \cos\frac{\pi}{3}$ $\qquad \cos x = \cos\frac{\pi}{3}$ $\qquad x = 2n\pi \pm \frac{\pi}{3}$

Example 5: Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$

Sol^n: $-\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right) = \tan\left(x + \frac{5\pi}{6}\right)$

$\therefore 2x = n\pi + x + \frac{5\pi}{6}$

$\therefore x = n\pi + \frac{5\pi}{6}$ $\quad n \in Z.$

Example 6 - Solve $\sin 2x - \sin 4x + \sin 6x = 0$

sol^n: $\sin 6x + \sin 2x - \sin 4x = 0$

$\because \sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

$\therefore \sin 6x + \sin 2x = 2 \cdot \sin \frac{6x+2x}{2} \cdot \cos \frac{6x-2x}{2} = 2 \cdot \sin 4x \cdot \cos 2x$

$\therefore \sin 6x + \sin 2x - \sin 4x = 2 \cdot \sin 4x \cdot \cos 2x - \sin 4x$

$= \sin 4x \, (2 \cdot \cos 2x - 1) = 0$

$\therefore \sin 4x = 0$ or $2 \cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2}$

$4x = n\pi$

$\cos 2x = \cos \frac{\pi}{3}$

$x = \frac{n\pi}{4}$

$2x = 2n\pi \pm \frac{\pi}{3}$

$x = n\pi \pm \frac{\pi}{6}$

Example-7: Solve $2 \cos^2 x + 3 \sin x = 0$

Solution: $2 \cos^2 x + 3 \sin x = 2(1 - \sin^2 x) + 3 \sin x$

$= 2 - 2 \sin^2 x + 3 \sin x$

$2 \times 2 = 4$
4×1
$-4 + 1 = -3$

$\therefore 2 \sin^2 x - 3 \sin x - 2 = 0$

$2 \sin^2 x - 4 \sin x + \sin x - 2 = 0$

$2 \sin x (\sin x - 2) + 1 (\sin x - 2) = 0$

$(2 \sin x + 1)(\sin x - 2) = 0$

$2 \sin x + 1 = 0$ $\quad$ $\sin x - 2 = 0$

$\therefore \sin x = -\frac{1}{2}$ $\quad$ $\sin x = 2$

$\sin x = 2$ (not possible) why?

$\sin x = -\frac{1}{2} = -\sin \frac{\pi}{6}$

$= \sin(\pi + \frac{\pi}{6})$

$= \sin \frac{7\pi}{6}$ $\quad$ $\therefore x = n\pi + (-1)^n \frac{7\pi}{6}$

Assignment · 4

Find the principal and general solutions of the followings:

1. $\tan x = \sqrt{3}$ 2. $\sec x = 2$ 3. $\cot x = -\sqrt{3}$

4. $\operatorname{cosec} x = -2$

Find the general solution for each of the following question:

5) $\cos 4x = \cos 2x$ 6) $\cos 3x + \cos x - \cos 2x = 0$

7) $\sin 2x + \cos x = 0$ 8) $\sec^2 2x = 1 - \tan 2x$

9) $\sin x + \sin 3x + \sin 5x = 0$

Things - To - Remember (Trigonometric functions - XI)

$1^\circ = 60'$ $\quad 1' = 60''$ $\quad \theta(\text{rad}) = \frac{l}{r}$ $\quad \pi(\text{rad}) = 180^\circ$

$\sin(2n\pi + x) = \sin x$ $\quad \cos(2n\pi + x) = \cos x$ $\quad n \in Z$

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

$\cos(-x) = \cos x$ $\quad \sin(-x) = -\sin x$ $\quad (a, b)$ $\quad a = \cos x$ $\quad b = \sin x$

$\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$ $\quad \cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ $\quad \sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$ $\quad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$\cot(A+B) = \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$ $\quad \cot(A-B) = \frac{\cot A \cdot \cot B - 1}{\cot B - \cot A}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$ $\quad \sin 2A = 2\sin A \cdot \cos B = \frac{2\tan A}{1 + \tan^2 A}$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$ $\quad \sin 3A = 3\sin A - 4\sin^3 A$ $\quad \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\cos A + \cos B = 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$ $\quad \cos A - \cos B = -2\sin\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$

$\sin A + \sin B = 2\sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$ $\quad \sin A - \sin B = 2\cos\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$ $\quad -2\sin A \cdot \sin B = \cos(A+B) - \cos(A-B)$

$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$ $\quad 2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

$\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$ $\quad 2\sin^2 A = 1 - \cos 2A$

$\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B)$ $\quad 2\cos^2 A = 1 + \cos 2A$

$\tan(45^\circ + \theta) = \frac{1 + \tan\theta}{1 - \tan\theta}$

$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$