

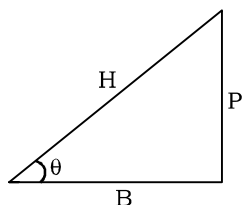
TRIGONOMETRY

TRIGONOMETRIC RATIOS & IDENTITIES

1. The meaning of Trigonometry

Tri Gon Metron
 ↓ ↓ ↓
 3 sides Measure

Hence, this particular branch in Mathematics was developed in ancient past to measure 3 sides, 3 angles and 6 elements of a triangle. In today's time—trigonometric functions are used in entirely different shapes. The 2 basic functions are sine and cosine of an angle in a right-angled triangle and there are 4 other derived functions.



$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\operatorname{cosec} \theta$
$\frac{P}{H}$	$\frac{B}{H}$	$\frac{P}{B}$	$\frac{B}{P}$	$\frac{H}{B}$	$\frac{H}{P}$

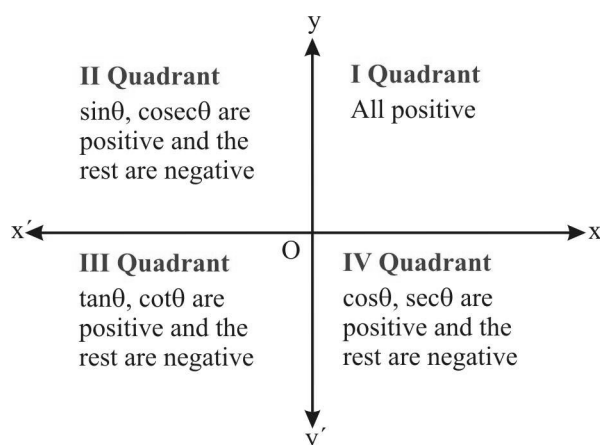
2. Basic Trigonometric Identities

- (a) $\sin^2\theta + \cos^2\theta = 1 : -1 \leq \sin\theta \leq 1; -1 \leq \cos\theta \leq 1 \quad \forall \theta \in \mathbb{R}$
- (b) $\sec^2\theta - \tan^2\theta = 1 : |\sec\theta| \geq 1 \quad \forall \theta \in \mathbb{R}$
- (c) $\operatorname{cosec}^2\theta - \cot^2\theta = 1 : |\operatorname{cosec}\theta| \geq 1 \quad \forall \theta \in \mathbb{R}$

Trigonometric Ratios of Standard Angles

T-Ratio ↓	Angle (θ)				
	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

The sign of the trigonometric ratios in different quadrants are as under :



3. Trigonometric Ratios of Allied Angles

Using trigonometric ratio of allied angles, we could find the trigonometric ratios of angles of any magnitude.

$$\sin(-\theta) = -\sin \theta \qquad \cos(-\theta) = \cos \theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan(-\theta) = -\tan \theta \qquad \cot(-\theta) = -\cot \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta \qquad \sec(-\theta) = \sec \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec} \theta \qquad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\sin(\pi - \theta) = \sin \theta \qquad \cos(\pi - \theta) = -\cos \theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta \qquad \cot\left(\frac{\pi}{2} + \theta\right) = -\tan \theta$$

$$\tan(\pi - \theta) = -\tan \theta \qquad \cot(\pi - \theta) = -\cot \theta$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec} \theta \qquad \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec \theta$$

$$\sec(\pi - \theta) = -\sec \theta \qquad \operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta$$

$$\sin(\pi + \theta) = -\sin \theta \qquad \cos(\pi + \theta) = -\cos \theta$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos \theta \qquad \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin \theta$$

$$\tan(\pi + \theta) = \tan \theta \qquad \cot(\pi + \theta) = \cot \theta$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot \theta \qquad \cot\left(\frac{3\pi}{2} - \theta\right) = \tan \theta$$

$$\sec(\pi + \theta) = -\sec \theta \qquad \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec} \theta$$

$$\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec} \theta \qquad \operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec \theta$$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta \qquad \cos\left(\frac{3\pi}{2} + \theta\right) = \sin \theta$$

$$\sin(2\pi - \theta) = -\sin \theta \qquad \cos(2\pi - \theta) = \cos \theta$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot \theta \qquad \cot\left(\frac{3\pi}{2} + \theta\right) = -\tan \theta$$

$$\tan(2\pi - \theta) = -\tan \theta \qquad \cot(2\pi - \theta) = -\cot \theta$$

$$\sec\left(\frac{3\pi}{2} + \theta\right) = \operatorname{cosec} \theta \qquad \operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec \theta$$

$$\sec(2\pi - \theta) = \sec \theta \qquad \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec} \theta$$

$$\sin(2\pi + \theta) = \sin \theta \qquad \cos(2\pi + \theta) = \cos \theta$$

$$\tan(2\pi + \theta) = \tan \theta \qquad \cot(2\pi + \theta) = \cot \theta$$

$$\sec(2\pi + \theta) = \sec \theta \qquad \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec} \theta$$

4. Trigonometric Functions of Sum or Difference of Two Angles

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(e) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(f) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(g) $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

(f) $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

(h) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A + B) \cdot \sin(A - B)$

(i) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)$

(j) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

5. Multiple Angles and Half Angles

(a) $\sin 2A = 2 \sin A \cos A$; $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$;

$2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta$, $2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta$

(c) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$; $\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$

(d) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$; $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

(e) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(f) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(g) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

6. Transformation of Products into Sum or Difference of Sines & Cosines

(a) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(b) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

(c) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(d) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

7. Factorisation of the Sum or Difference of Two Sines or Cosines

(a) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(b) $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

(c) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(d) $\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$

8. Important Trigonometric Ratios

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in \mathbb{Z}$

(b) $\sin 15^\circ$ or $\sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos \frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin \frac{5\pi}{12}$;

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$;

$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^\circ$

(c) $\sin \frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ &

$\cos 36^\circ$ or $\cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

9. Conditional Identities

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$

(vii) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

10. Range of Trigonometric Expression

$E = a \sin \theta + b \cos \theta$

$E = \sqrt{a^2 + b^2} \sin(\theta + \alpha)$, (where $\tan \alpha = \frac{b}{a}$)

$E = \sqrt{a^2 + b^2} \cos(\theta - \beta)$, (where $\tan \beta = \frac{a}{b}$)

Hence for any real value of θ , $-\sqrt{a^2 + b^2} \leq E \leq \sqrt{a^2 + b^2}$

11. Sine and Cosine Series

(a) $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + \overline{n-1}\beta)$

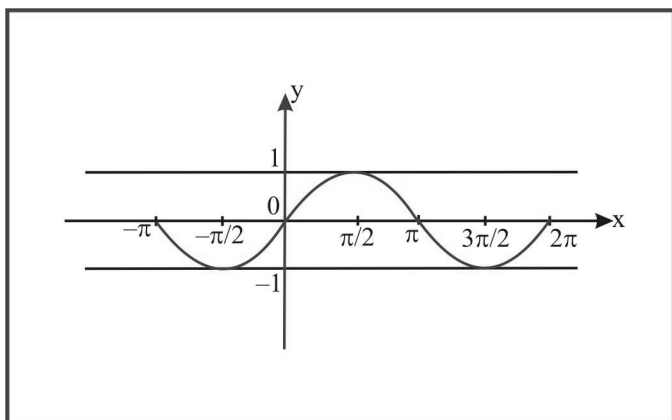
$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin \left(\alpha + \frac{n-1}{2}\beta \right)$$

(b) $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots + \cos (\alpha + \overline{n-1}\beta)$

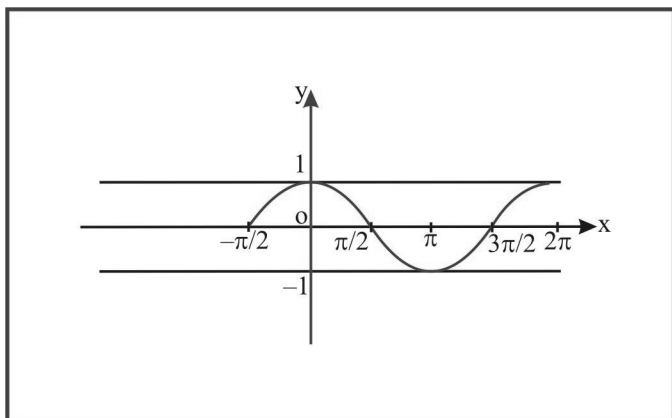
$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left(\alpha + \frac{n-1}{2}\beta \right)$$

12. Graphs of Trigonometric Functions

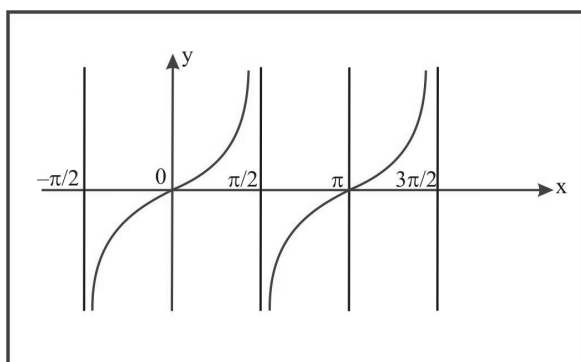
(a) $y = \sin x,$
 $x \in \mathbb{R}; y \in [-1, 1]$



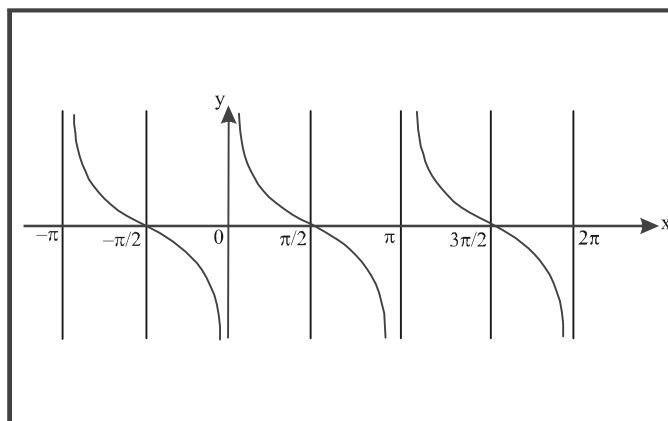
(b) $y = \cos x,$
 $x \in \mathbb{R}; y \in [-1, 1]$



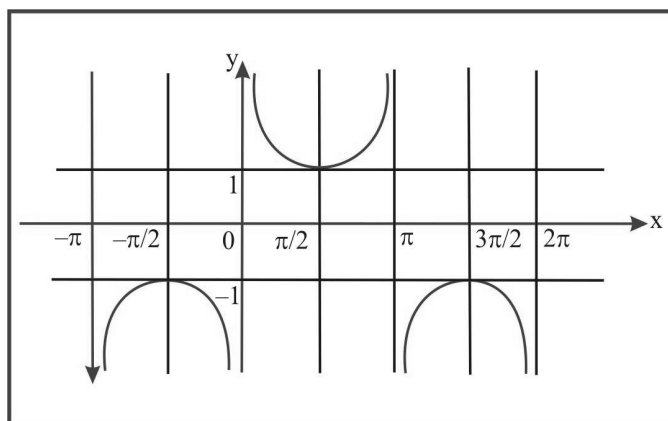
(c) $y = \tan x,$
 $x \in \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \right\}; y \in \mathbb{R}$



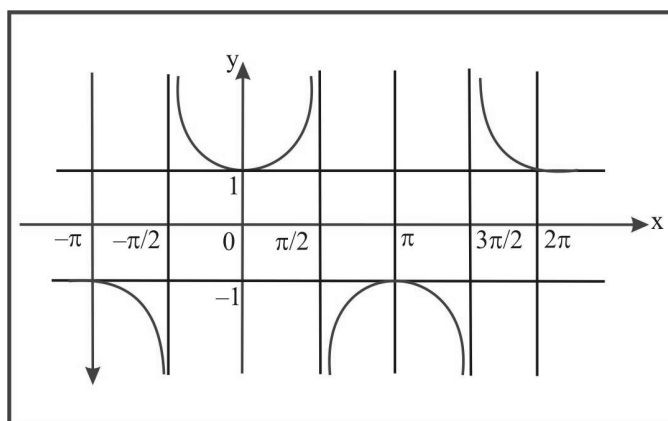
(d) $y = \cot x,$
 $x \in \mathbb{R} - \{n\pi; n \in \mathbb{Z}\}; y \in \mathbb{R}$



(e) $y = \operatorname{cosec} x,$
 $x \in \mathbb{R} - \{n\pi; n \in \mathbb{Z}\}; y \in (-\infty, -1] \cup [1, \infty)$



(f) $y = \sec x,$
 $x \in \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \right\}; y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC EQUATIONS

13. Trigonometric Equations

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations.

e.g., $\cos \theta = 0$, $\cos^2 \theta - 4 \cos \theta = 1$.

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g., $\sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \dots$

Thus, the trigonometric equation may have infinite number of solutions and can be classified as :

- (i) Principal solution
- (ii) General solution

14. General Solution

Since, trigonometric functions are periodic, a solution generalised by means of periodicity of the trigonometrical functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

14.1 Results

1. $\sin \theta = 0 \Leftrightarrow \theta = n \pi$
2. $\cos \theta = 0 \Leftrightarrow \theta = (2n + 1) \frac{\pi}{2}$
3. $\tan \theta = 0 \Leftrightarrow \theta = n \pi$

4. $\sin \theta = \sin \alpha \Leftrightarrow \theta = n \pi + (-1)^n \alpha$, where

$$\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

5. $\cos \theta = \cos \alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$.

6. $\tan \theta = \tan \alpha \Leftrightarrow \theta = n \pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

7. $\sin^2 \theta = \sin^2 \alpha \Leftrightarrow \theta = n \pi \pm \alpha$.

8. $\cos^2 \theta = \cos^2 \alpha \Leftrightarrow \theta = n \pi \pm \alpha$.

9. $\tan^2 \theta = \tan^2 \alpha \Leftrightarrow \theta = n \pi \pm \alpha$.

10. $\sin \theta = 1 \Leftrightarrow \theta = (4n + 1) \frac{\pi}{2}$.

11. $\cos \theta = 1 \Leftrightarrow \theta = 2n \pi$.

12. $\cos \theta = -1 \Leftrightarrow \theta = (2n + 1) \pi$.

13. $\sin \theta = \sin \alpha$ and $\cos \theta = \cos \alpha \Leftrightarrow \theta = 2n \pi + \alpha$.



1. Every where in this chapter 'n' is taken as an integer, if not stated otherwise.
2. The general solution should be given unless the solution is required in a specified interval or range.
3. α is taken as the principal value of the angle. (i.e., Numerically least angle is called the principal value).