Chapter 8 - Binomial Theorem (XI)

Introduction

An algebraic expressions consisting of two terms is called a binomial expression, for example, $1+x$, $x+3a$, $x-\frac{1}{x}$ etc. are all binomial expressions.

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a+b$ and $a-b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100-2)^2$, $(999)^3 = (1000-1)^3$ etc. However, for highers powers like $(98)^5$, $(101)^6$ etc, the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem.
It gives an easier way to expand $(a+b)^n$, where n is an integer or a rational number.

Binomial Theorem for Positive Integral Indices

If n is a positive integer, then

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} \cdot a^2 + \dots$$
$$+ {}^nC_r\, x^{n-r} a^r + \dots + {}^nC_n\, a^n$$

Proof is obtained by applying the principle of mathematical induction.

$${}^nC_r = \frac{n!}{r!\,(n-r)!} \qquad 0 \leq r \leq n.$$

Remember

(1) In the expansion of $(x+a)^n$, there are $(n+1)$ terms, beginning with x^n and ending with a^n.

(2) As we move from each term to the next, powers of x decreases by 1 and powers of a increases by 1. Therefore, in each term, the exponent of x and a add up to n.

(3) The coefficient ${}^nC_0, {}^nC_1, {}^nC_2, \dots {}^nC_n$ are called binomial coefficients. In shorts, they are written as $C_0, C_1, C_2, \dots C_n$.

(4) In the binomial expansion, the terms equidistant from the beginning and the end have numerically equal coefficients. It is so because ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}$, ... ${}^nC_r = {}^nC_{n-r}$ and so on.

The coefficients of $(x+a)^1$ are 1, 1

" $(x+a)^2$ are 1, 2, 1

" $(x+a)^3$ are 1, 3, 3, 1

" $(x+a)^4$ are 1, 4, 6, 4, 1

(5) The binomial coefficient can be remembered by Pascal's Triangle..

The Pascal's triangle can now be rewritten as

Index	Coefficients
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

Pascal's triangle

(6) Replacing x by 1 and a by x in $(x+a)^n$, we get

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^r .$$

Remember, ${}^nC_0 = 1$

(i) Replacing x by 1 in $(1+x)^n$

$$(1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_r + \dots + {}^nC_n = 2^n$$

(ii) Putting $x = a = 1$ in $(x+a)^n$

$$(1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_r + \dots + {}^nC_n = 2^n$$

Hence, the sum of all the binomial coefficients $= 2^n$.

7) Expansion of $(1-x)^n$

Changing x to $-x$ in $(1+x)^n$,

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + {}^nC_r (-x)^r + \dots (-x)^n.$$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^r$$

8) mth term from the end:

In the binomial expansion of $(x+a)^n$, the mth term from the end

$= \{(n+1) - m + 1\}$th term from the beginning

$= \{n - m + 2\}$th term from the beginning.

Solved Examples

(1) Find the number of terms in the expansions of the following :
(i) $[(2x+3y)^2]^5$ (ii) $[(3x+y^2)^9]^4$.

Sol^n: (i) $[(2x+3y)^2]^5 = (2x+3y)^{10}$.

$\therefore$ Number of term in expansion $= 10+1 = 11$

(ii) $[(3x+y^2)^9]^4 = (3x+y^2)^{36}$

No. of term in expansion $= 36+1 = 37$.

(2) Expand $\left(x^2+\frac{3}{x}\right)^4$, $x \neq 0$.

Sol^n: Using binomial theorem,

$$\left(x^2+\frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3\left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4\cdot x^6 \cdot \frac{3}{x} + 6x^4 \cdot \frac{9}{x^2} + 4\cdot x^2 \cdot \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$$

(3) Using binomial theorem, find the value of $(a+b)^4 - (a-b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

Sol^n: $(a+b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ ——(1)

$(a-b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3(-b) + {}^4C_2 a^2(-b)^2 + {}^4C_3 a(-b)^3 + {}^4C_4 (-b)^4$
$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$(a+b)^4 - (a-b)^4 = 4a^3b + 4ab^3 + 4a^3b + 4ab^3 = 8ab(a^2+b^2)$.

(4) Compute $(98)^5$

Sol^n $98 = 100 - 2$ $\quad (98)^5 = (100-2)^5$

$$\therefore (100-2)^5 = {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (-2) + {}^5C_2 (100)^3 (-2)^2$$

$$+ {}^5C_3 (100)^2 (-2)^3 + {}^5C_4 (100)(-2)^4 + {}^5C_5 (-2)^5$$

$$= (10)^{10} - 5 \times 10^8 \times 2 + 10 \times 10^6 \times 4 - 10 \times 10^4 \times 8$$
$$+ 5 \times 100 \times 16 - 32$$

$$= 1004\,000\,8000 - 1000\,800\,032$$
$$= 9039207968 .$$

5) Which is larger $(1.01)^{1000000}$ or $10,000$?

Sol^n: $(1.01)^{1000\,000} = (1+0.01)^{1000\,000}$

$$= {}^{1000\,000}C_0 + {}^{1000\,000}C_1 (0.01) + \text{other positive term.}$$
$$= 1 + 1000\,000 \times 0.01 + \text{other positive term.}$$

$$> 10,000 .$$

6) Using binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25.

Sol^n: For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder.

Thus, in order to show that $6^n - 5n$ leaves remainder 1 when divided by 25, we prove that $6^n - 5n = 25k + 1$, where k is some natural number.

We have $(1+a)^n = {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + \dots + {}^nC_n a^n$

$a = 5$, $(1+5)^n = {}^nC_0 + {}^nC_1 5 + {}^nC_2 5^2 + \dots {}^nC_n 5^n$

$\therefore (6)^n = 1 + 5n + 5^2\,{}^nC_2 + 5^3\,{}^nC_3 + \dots + 5^n$

$6^n - 5n = 1 + 5^2({}^nC_2 + 5\,{}^nC_3 + \dots 5^{n-2})$

$6^n - 5n = 25k + 1$ where $k = {}^nC_2 + 5\cdot{}^nC_3 + \dots + 5^{n-2}$

This shows that when divided by 25, $6^n - 5n$ leaves remainder 1.

7) If P be the sum of odd terms and Q the sum of even terms in the expansion of $(x+a)^n$, prove that

$$P^2 - Q^2 = (x^2 - a^2)^n \qquad 4PQ = (x+a)^{2n} - (x-a)^{2n}.$$

Sol^n: $(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_n a^n$

$(x+a)^n = P + Q$ ——— (1)

changing a to $-a$, we get

$(x-a)^n = x^n + {}^nC_1 x^{n-1}(-a) + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3}(-a)^3$

$= x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 - ({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \dots)$

$(x-a)^n = P - Q$ ——— (2)

Multiplying (1) & (2) $(x+a)^n (x-a)^n = (P+Q)(P-Q)$

$\Rightarrow (x^2 - a^2)^n = P^2 - Q^2$

$4PQ = (P+Q)^2 - (P-Q)^2 = \{(x+a)^{2n} - (x-a)^{2n}\}$

General and Middle Term in Binomial Expansion

We know that, if n is a positive integers and if x and a are any two numbers, then

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_r x^{n-r} \cdot a^r + \dots + {}^nC_n a^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} \cdot a^r$$

$(r+1)$th term $= {}^nC_r x^{n-r} a^r \quad 0 \le r \le n.$

The $(r+1)$th term in the expansion of $(x+a)^n$ is called its general term. Denoted by T_{r+1} or t_{r+1}

Thus, $\boxed{T_{r+1} = {}^nC_r x^{n-r} a^r ; \; 0 \le r \le n}$ $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Remember,

(i) In the expansion of $(x-a)^n$, $T_{r+1} = (-1)^r \, {}^nC_r \, x^{n-r} \cdot a^r$.

(ii) In the expansion of $(1+x)^n$, $T_{r+1} = {}^nC_r \, x^r$

(iii) In the expansion of $(1-x)^n$, $T_{r+1} = (-1)^r \, {}^nC_r x^r$.

Middle Term

There are $(n+1)$ terms in the expansion of $(x+a)^n$.

(i) When n is even, middle term is $\left(\frac{n}{2}+1\right)$

(ii) When n is odd, there are two middle terms $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th term.

Greatest Binomial Coefficients

Middle term always carries greatest binomial coefficients.

(i) If n is even, then middle term $T_{\left(\frac{n}{2}+1\right)}$ has greatest binomial coefficient ${}^{n}C_{\frac{n}{2}}$.

(ii) If n is odd, then middle terms $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$ has greatest binomial coefficients

${}^{n}C_{\left(\frac{n-1}{2}\right)}$ and ${}^{n}C_{\left(\frac{n+1}{2}\right)}$

Greatest term in the expansion of $(x+a)^n$, $n \in N$.

$$\frac{T_{r+1}}{T_r} = \frac{{}^{n}C_r\, x^{n-r} \cdot a^r}{{}^{n}C_{r-1}\, x^{n-r+1}\, a^{r-1}} = \frac{{}^{n}C_r}{{}^{n}C_{r-1}} \cdot \frac{a}{x}$$

$$= \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(r-1)!\,(n-r+1)!}} \times \frac{a}{x} = \frac{n-r+1}{r} \cdot \frac{a}{x}$$

Now $T_{r+1} > T_r \Rightarrow \frac{n-r+1}{r} \cdot \frac{a}{x} > 1 \Rightarrow r < \frac{(n+1)a}{(x+a)}$

Case I: When $\frac{(n+1)}{x+a}a$ is an integer

Let $\frac{(n+1)}{x+a}a = k$ when $r = k$, $T_{k+1} = T_k$ and these are two greatest terms in the expansion of $(x+a)^n$.

Case II: When $\frac{(n+1)a}{x+a}$ is a fraction.

Let m be the integral part of $\frac{(n+1)a}{x+a}$. T_{m+1} is the greatest term in the expansion of $(x+a)^n$.

Solved Examples

(1) Write the general term in the expansion of $(x^2-y)^6$.

Sol^n: $T_{r+1} = {}^nC_r\, x^{n-r} a^r$

$T_{r+1} = {}^6C_r\, x^{6-r} \cdot a^r$

$x = x^2 \quad a = -y \;\; \therefore\; T_{r+1} = {}^6C_r (x^2)^{6-r} \cdot (-y)^r$

$T_{r+1} = {}^6C_r\, x^{12-2r} \cdot (-y)^r$.

(2) Find the 4th term in the expansion of $(x-2y)^{12}$

Sol^n: $T_{r+1} = {}^nC_r\, x^{n-r} \cdot a^r$

put $r = 3$

$T_4 = {}^{12}C_3\, x^{12-3} \cdot (-2y)^3$

$T_4 = {}^{12}C_3\, x^9 (-8) y^3 = \dfrac{(12)!}{3!\,9!} (-8) \times x^9 y^3$

$= \dfrac{\cancel{12}^{2} \times 11 \times 10}{\cancel{3} \times \cancel{2}} \times (-8)\, x^9 y^3 = -1760\, x^9 y^3$

(3) Find a if the 17th and 18th terms of the expansion $(2+a)^{50}$ are equal.

Sol^n: $T_{r+1} = {}^nC_r\, x^{n-r} \cdot a^r$

$T_{17} = {}^{50}C_{16}\, x^{50-16} a^{16} = {}^{50}C_{16}\, 2^{34} a^{16}$

Similarly $T_{18} = {}^{50}C_{17}\, 2^{33} a^{17}$ Given $T_{17} = T_{18}$

${}^{50}C_{16}\, 2^{34} a^{16} = {}^{50}C_{17}\, 2^{33} a^{17} \Rightarrow a = \dfrac{{}^{50}C_{16}}{{}^{50}C_{17}} \times 2 = 1.$

(4) If the rth term in the expansion of $\left(\frac{x}{3}-\frac{2}{x^2}\right)^{10}$ contains x^4, then find the value of r.

Sol^n: $x = \frac{x}{3}$ $a = -\frac{2}{x^2}$

$T_{r+1} = {}^nC_r\, x^{n-r} \cdot a^r$, $T_r = {}^nC_{r-1}\, x^{n-r+1} \cdot a^{r-1}$

$\therefore T_r = {}^{10}C_{r-1} \left(\frac{x}{3}\right)^{10-r+1} \cdot \left(\frac{-2}{x^2}\right)^{r-1}$

$T_r = {}^{10}C_{r-1} \left(\frac{x}{3}\right)^{11-r} \frac{(-2)^{r-1}}{x^{2r-2}}$

$= {}^{10}C_{r-1} \left(\frac{1}{3}\right)^{11-r} \times (-2)^{r-1} \cdot x^{11-r-2r+2}$

$= {}^{10}C_{r-1} \left(\frac{1}{3}\right)^{11-r} (-2)^{r-1}\, x^{13-3r}$.

T_r will contain x^4, if $13-3r = 4 \Rightarrow r = 3$.

5) Find the middle terms in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$.

Sol^n: Here $n = 11$ $\therefore$ The no. of terms $= 11+1 = 12$ (even)

There are two middle term: $\frac{11+1}{2}$ and $\frac{11+3}{2}$ th

i.e. 6^{th} & 7^{th} Term.

$x = x^2$ $a = \frac{1}{x}$ $T_6 = {}^{11}C_5 (x^2)^{11-5} \left(\frac{1}{x}\right)^5 = {}^{11}C_5\, x^{12} \cdot \frac{1}{x^5}$

$T_6 = {}^{11}C_5 x^7 = \frac{11!}{5!(11-5)!} x^7 = \frac{11!}{5!(6)!} x^7 = 462 x^7$

$T_7 = {}^{11}C_6 (x^2)^{11-6} \left(\frac{1}{x}\right)^6 = {}^{11}C_6\, x^4 = \frac{11!}{6!(11-6)!} x^4 = \frac{11!}{6!\,5!} x^4 = 462 x^4$.

(6) Find the middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$.

Sol^n : $n=10$, no. of term $=10+1=11$ (odd)

There is only one middle term $\left(\frac{n}{2}+1\right)$th

$$=\frac{10}{2}+1=6\text{th}$$

$$T_6=T_{5+1}={}^{10}C_5\left(\frac{2x^2}{3}\right)^{10-5}\left(\frac{3}{2x^2}\right)^5$$

$$={}^{10}C_5\,\frac{2^5x^{10}}{3^5}\times\frac{3^5}{2^5\cdot x^{10}}={}^{10}C_5$$

$$=\frac{(10)!}{5!\,5!}=252.$$

(7) Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}2^n x^n$, where n is a positive integer.

Sol^n : No. of terms in expansion $=(2n+1)$ odd.

$\therefore$ There is only one middle term $\left(\frac{2n}{2}+1\right)$th or $(n+1)$th

$$T_{n+1}={}^{2n}C_n\,x^n=\frac{(2n)!}{n!\,n!}x^n$$

$$=\frac{2n(2n-1)(2n-2)\cdots4\cdot3\cdot2\cdot1\;x^n}{(n!)^2}$$

$$=\frac{[2n(2n-2)(2n-4)\cdots4\cdot2][(2n-1)(2n-3)\cdots3\cdot1]}{(n!)^2}x^n$$

$$=\frac{2^n[n(n-1)(n-2)\cdots2\cdot1][(2n-1)\cdots3\cdot1]}{(n!)^2}x^n$$

$$=\frac{2^n\cdot\cancel{n!}\,[1\cdot3\cdot5\cdots(2n-1)]}{(n!)^{\cancel{2}}}x^n$$

$$=\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}2^nx^n.$$

8) Find the coefficient of x^6y^3 in the expansion of $(x+2y)^9$.

Soln: $T_{r+1} = {}^9C_r\, x^{9-r} \cdot (2y)^r = {}^9C_r\, 2^r\, x^{9-r}\, y^r$

Comparing the indices of x and y in x^6y^3 in T_{r+1}

we get $r = 3$

$\therefore$ coefficient of x^6y^3 is $= {}^9C_3\, 2^3 = \dfrac{9!}{3!\,6!}\, 2^3 = 672$.

(9) The second, third and fourth terms in the binomial expansion $(x+a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Soln: Given $T_2 = 240 \quad T_3 = 720 \quad T_4 = 1080$

$T_2 = T_{1+1} = {}^nC_1\, x^{n-1} \cdot a = 240$ —— (1)

$T_3 = T_{2+1} = {}^nC_2\, x^{n-2}\, a^2 = 720$ —— (2)

$T_4 = T_{3+1} = {}^nC_3\, x^{n-3}\, a^3 = 1080$ —— (3)

Dividing $\dfrac{T_3}{T_2}$, $\dfrac{{}^nC_2\, x^{n-2}\, a^2}{{}^nC_1\, x^{n-1}\, a} = \dfrac{\cancel{720}^{\,3}}{\cancel{240}_{\,1}} \Rightarrow \dfrac{(n-1)!}{(n-2)!} \cdot \dfrac{a}{x} = 6$

$\Rightarrow \dfrac{a}{x} = \dfrac{6}{n-1}$

Dividing $\dfrac{T_4}{T_3}$, $\dfrac{a}{x} = \dfrac{9}{2(n-2)}$

$\dfrac{6}{n-1} = \dfrac{9}{2(n-2)}$ thus $n = 5$.

${}^nC_1\, x^4\, a = 240$

$5x^4 a = 240$

$\dfrac{a}{x} = \dfrac{6}{5-1} = \dfrac{3}{2}$

Solving $x = 2$

$a = 3$.

(10) The coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio $1:7:42$. Find n.

Sol^n: Suppose the three consecutive terms in the expansion of $(1+a)^n$ are $(r-1)$th, rth and $(r+1)$th term.

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7} \quad \text{i.e.} \quad n-8r+9=0$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42} \quad \text{i.e.} \quad n-7r+1=0$$

Solving we get $n=55$.

11) Find the term independ of x in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$.

Sol^n: $T_{r+1} = {}^nC_r\, x^{n-r} \cdot c^r$

$$\therefore T_{r+1} = {}^6C_r\left(\frac{3}{2}x^2\right)^{6-r}\left(-\frac{1}{3x}\right)^r$$

$$= {}^6C_r\left(\frac{3}{2}\right)^{6-r}(x^2)^{6-r}(-1)^r\frac{1}{3^r}\left(\frac{1}{x}\right)^r$$

$$= (-1)^r\, {}^6C_r\,\frac{(3)^{6-2r}}{(2)^{6-r}}\, x^{12-3r}$$

The term will be independent of x if the index of x is zero, i.e. $12-3r=0 \Rightarrow r=4$.

Hence 5th term is independent of x and is given by $(-1)^4\, {}^6C_4\,\frac{(3)^{6-8}}{(2)^{6-4}} = \frac{5}{12}$

(12) If the coefficients of a^{r-1}, a^r and a^{r+1} in the expansion of $(1+a)^n$ are in arithmetic progress, prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

Sol^n: $T_{r+1} = {}^nC_r a^r$

$T_r = {}^nC_{r-1} a^{r-1}$

$T_{r+2} = {}^nC_{r+1} a^{r+1}$

Since these coefficients ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ are in arithmetic progression,

$\therefore\ {}^nC_r - {}^nC_{r-1} = {}^nC_{r+1} - {}^nC_r \quad \therefore\ 2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

Thus, $2 \times \dfrac{\cancel{n!}}{r!(n-r)!} = \dfrac{\cancel{n!}}{(r-1)!(n-r+1)!} + \dfrac{\cancel{n!}}{(r+1)!(n-r-1)!}$

$$\frac{2}{r\cancel{(r-1)!}(n-r)\cancel{(n-r-1)!}} = \frac{1}{\cancel{(r-1)!}(n-r+1)(n-r)\cancel{(n-r-1)!}} + \frac{1}{(r+1)r\cancel{(r-1)!}\cancel{(n-r-1)!}}$$

$$\frac{2}{r(n-r)} = \frac{1}{(n-r+1)(n-r)} + \frac{1}{(r+1)r}$$

$$\frac{2}{\cancel{r}\cancel{(n-r)}} = \frac{r(r+1) + (n-r+1)(n-r)}{\cancel{(n-r)}(n-r+1)(r+1)\cancel{r}}$$

$$2(r+1)(n-r+1) = r(r+1) + (n-r+1)(n-r)$$

$$2[rn - r^2 + \cancel{r} + n - \cancel{r} + 1] = r^2 + \cancel{r} + n^2 - nr + n - rn + r^2 - \cancel{r}$$

$$2rn - 2r^2 + 2n + 2 = 2r^2 + n^2 - 2rn + n$$

$$n^2 - 4rn + 4r^2 - n - 2 = 0$$

$$n^2 - n(4r+1) + 4r^2 - 2 = 0.$$

13) Show that the coefficients of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.

Sol^n: As $2n$ is even

The expansion has $(2n+1)$ terms (odd)

There is only one middle term which is $\left(\frac{2n}{2}+1\right)^{th}$

ie. $(n+1)^{th}$

$T_{n+1} = {}^{2n}C_n \, x^n$. The coefficient of x^n is ${}^{2n}C_n$.

Similarly $2n-1$ is odd

Expansion has $2n-1+1 = 2n$ terms (even)

It has two middle term $\left(\frac{2n-1+1}{2}\right)^{th}$ & $\left(\frac{2n-1+1}{2}+1\right)^{th}$

ie. n^{th} and $(n+1)^{th}$ term.

$T_n = {}^{2n-1}C_{n-1} \, x^{n-1}$ $\quad T_{n+1} = {}^{2n-1}C_n \cdot x^n$

Now ${}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n \quad \left[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r\right]$

(14) Find the coefficient of a^4 in the product $(1+2a)^4(2-a)^5$ using binomial theorem. ANS: $-438a^4$.

[Hint... First expand each of the factors using binomial theorem and then multiply.
No need complete multiplication. Write only those terms which involve a^4.

(15) Find the r^{th} term from the end in the expansion of $(x+a)^n$.

Sol^n: r^{th} term from the end will be term number $(n+1)-(r-1) = (n-r+2)$ of the expansion.

And the $(n-r+2)^{th}$ term is ${}^nC_{n-r+1}\, x^{r-1} \cdot a^{n-r+1}$.

16) Find the term independent of x in the expansion of $\left(\sqrt[3]{x} + \frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x>0$.

Sol^n: $T_{r+1} = {}^{18}C_r\, (x)^{\frac{18-r}{3}} \cdot \left(\frac{1}{2(x)^{1/3}}\right)^r$

$$= {}^{18}C_r \cdot \frac{(x)^{\frac{18-r}{3}}}{2^r\, x^{r/3}} = {}^{18}C_r\, \frac{1}{2^r}\, x^{\frac{18-2r}{3}}$$

To find term independent of x, $\frac{18-2r}{3} = 0$, $r = 9$.

$\therefore$ The required term is ${}^{18}C_9\, \frac{1}{2^9}$.

17) The sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^m$, $x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3. [Ans: -5940].

18) If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r. [Ans $r=14$]