Chap 4 : Principle of Mathematical Induction (XI)

Introduction

The word 'Induction' means the generalisation from particular facts or cases in a simple language.
In algebra, there are certain statements and results which are formulated in terms of n, where n is a positive integer.
In order to prove such statements, we use the well suitable principle known as Principle of Mathematical Induction, which is based on specific technique.

Example, (i) Eight is divisible by two.
(ii) Any number divisible by two is an even number
(iii) Eight is an even number.

If statements (i) and (ii) are true, then the truth of (iii) is established.

Motivation

A set of thin rectangular tiles, which are placed on one side. When the first tile is pushed in a particular direction, all the tiles will fall.

(i) The first tile falls
(ii) When any tile falls, the succeeding tile also falls.

This is the underlying principle of Mathematical Induction.

Illustration : Suppose we wish to find the formula for the sum of positive integers $1, 2, 3, \ldots, n$.

$1+2+3+\ldots+n = \frac{n(n+1)}{2}$ is the correct one

How can this formula actually be proved?
This formula is proved by method of mathematical induction.

The Principle of Mathematical Induction

Let $P(n)$ be a statement of a theorem for all $n \in N$.

Step (I) : Verify the theorem for $n = 1$ i.e. Verify $P(1)$.

Step (II) : Suppose $P(n)$ is true for $n = k$, where $1 \le k < n$.

Step (III) : Verify that $P(n)$ is true for $n = k+1$

Then $P(n)$ is true for all $n \in N$.

Example 1 : For all $n \ge 1$, prove that

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution : Let the given statement be $P(n)$, i.e.

$$P(n) : 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$ $\quad P(1) = \frac{1(1+1)(2 \times 1 + 1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$ which is true.

Suppose $P(n)$ is true for $n = k$, i.e.

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$ where $1 \le k < n$.

Now we shall prove that $P(k+1)$ is also true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

$6 \times 2 = 12 = 4 \times 3$

$2k^2 + 7k + 6 = 2k^2 + 4k + 3k + 6$
$= 2k(k+2) + 3(k+2)$
$= (k+2)(2k+3)$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)\{(2(k+1)+1\}}{6}$$

Thus $P(k+1)$ is true whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers N.

Example 2: Prove that $2^n > n$ for all positive integers n.

Solution: Let $P(n): 2^n > n$

When $n = 1$ $P(1): 2^1 > 1$ $P(1)$ is true.

Assume that $P(k)$ is true for any positive integers k, i.e.
$2^k > k$

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true

Multiplying both sides by 2, $2 \cdot 2^k > 2k$
$2^{k+1} > 2k = k+k > k+1$

$\therefore$ $P(k+1)$ is true when $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true.

Example 3: for all $n \geq 1$, prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Soln: $P(n): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$P(1): \frac{1}{1\cdot 2} = \frac{1}{1+1} = \frac{1}{2}$ $P(1)$ is true.

Assume that $P(k)$ is true for some natural numbers k

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Now we shall prove that $P(k+1)$ is true whenever $P(k)$ is true.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{(k+1)+1}$$

Thus $P(k+1)$ is true whenever $P(k)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers.

Example 4: For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution: Let $P(n)$: $7^n - 3^n$ is divisible by 4 :

$P(1)$: $7^1 - 3^1 = 4$ which is divisible by 4. $P(1)$ is true.

Let $P(k)$ be true for some natural number k

i.e. $P(k) = 7^k - 3^k$ is divisible 4.

We can write $7^k - 3^k = 4d$ where $d \in \mathbb{N}$.

Now, we shall prove that $P(k+1)$ is true whenever $P(k)$ is true.

$$\begin{aligned}
\text{Now, } 7^{k+1} - 3^{k+1} &= 7^{(k+1)} - 7 \cdot 3^k + 7 \cdot 3^k - 3^{(k+1)} \\
&= 7(7^k - 3^k) + (7-3)3^k \\
&= 7(4d) + 4 \cdot 3^k \\
&= 4[7d + 3^k].
\end{aligned}$$

We see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4.

Thus $P(k+1)$ is true when $P(k)$ is true. Therefore, by principle of mathematical induction the statement is true for every positive integer n.

Example 5: Prove that $(1+x)^n \geq (1+nx)$, for all natural number n, where $x > -1$.

Solution: Let $P(n)$ be the given statement

$P(n)$: $(1+x)^n \geq (1+nx)$, for $x > -1$.

When $n = 1$ $P(1)$: $(1+x)^1 \geq (1+1 \cdot x)$

$(1+x) \geq (1+x)$ for $x > -1$. $P(1)$ is true.

Assume that $P(k)$: $(1+x)^k \geq (1+kx)$, $x > -1$ is true.

Now we shall prove that $P(K+1)$ is true for $x \geq -1$ whenever $P(K)$ is true:

Consider the identity $(1+x)^{K+1} = (1+x)^K \cdot (1+x)$

Given $x > -1$, so $(x+1) > 0$

Therefore, using $(1+x)^K \geq (1+Kx)$

$$(1+x)^K \cdot (1+x) \geq (1+Kx)\cdot(1+x)$$
$$(1+x)^{K+1} \geq (1+x+Kx+Kx^2)$$

Here $x^2 \geq 0$ so that $Kx^2 \geq 0$, therefore $(1+x+Kx+Kx^2) \geq (1+x+Kx)$

We obtain, $(1+x)^{K+1} \geq (1+x+Kx)$

i.e. $(1+x)^{K+1} \geq [1+(1+K)x]$

Thus, $P(K+1)$ is true whenever $P(K)$ is true.
Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers.

<u>Example 6</u>: Prove that $2\cdot7^n + 3\cdot5^n - 5$ is divisible by 24, for all $n \in N$

Solution: Let $P(n)$ be defined as

$$P(n): 2\cdot7^n + 3\cdot5^n - 5 \text{ is divisible by } 24$$

for $n=1$, $2\cdot7^1 + 3\cdot5^1 - 5 = 14+15-5 = 24$ which is divisible by 24. Hence $P(1)$ is true.

Now assume that $P(K)$ is true.

i.e. $P(K): 2\cdot7^K + 3\cdot5^K - 5 = 24q$ where $q \in N$.

Now, we shall prove that $P(k+1)$ is true whenever $P(k)$ is true.

$$2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5 = 2\cdot 7^{k}\cdot 7^{1} + 3\cdot 5^{k}\cdot 5^{1} - 5$$
$$= 7[2\cdot 7^{k} + 3\cdot 5^{k} - 5 - 3\cdot 5^{k} + 5] + 3\cdot 5^{k}\cdot 5^{1} - 5$$
$$= 7[24q - 3\cdot 5^{k} + 5] + 15\cdot 5^{k} - 5$$
$$= 7\times 24q - 21\cdot 5^{k} + 35 + 15\cdot 5^{k} - 5$$
$$= 7\times 24q - 6\cdot 5^{k} + 30$$
$$= 7\times 24q - 6\,(5^{k} - 5)$$
$$= 7\times 24q - 6(4p) \quad [\text{Since } 5^{k}-5 \text{ is multiple of } 4]$$
$$= 7\times 24q - 24p$$
$$= 24\,(7q - p)$$
$$= 24\times r \quad [\, r = 7q - p, \text{ is some natural number}]$$

Thus, $2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5$ is divisible by 24.
Thus $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 7: Prove that $1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}$, $n \in N$.

Solution: Let $P(n)$ be defined as

$P(n): 1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}$, $n \in N$.

When $n = 1$ $\quad 1 > \frac{1}{3}$. $P(1)$ is true for $n = 1$.

Assume that $P(k)$ is true.

$P(k): 1^2 + 2^2 + \dots + k^2 > \frac{k^3}{3}$

We shall now prove that $P(K+1)$ is true whenever $P(K)$ is true.

We have $1^2+2^2+3^2+\dots+k^2+(k+1)^2$

$$= (1^2+2^2+3^2+\dots k^2)+(k+1)^2 > \frac{k^3}{3}+(k+1)^2$$

$$= \frac{1}{3}[k^3+3k^2+6k+3]$$

$$= \frac{1}{3}[k^3+3k^2+3k+3k+2+1]$$

[using identity $(a+b)^3 = a^3+b^3+3a^2b+3ab^2$]

$$= \frac{1}{3}[(k^3+3k^2+3k+1)+3k+2]$$

$$= \frac{1}{3}[(k+1)^3+3k+2] > \frac{1}{3}(k+1)^3$$

Therefore, $P(K+1)$ is true whenever $P(k)$ is true.
Hence, by mathematical induction $P(n)$ is true for all $n \in N$.

Example 8: Prove that rule of exponents $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.

Solution: Let $P(n)$: $(ab)^n = a^n b^n$

$P(1)$: $(ab)^1 = a^1 \cdot b^1$ Hence $P(1)$ is true.

Let $P(K)$: $(ab)^k = a^k \cdot b^k$ be true

Now, we shall prove that $P(K+1)$ is true whenever $P(k)$ is true.

$$(ab)^{k+1} = (ab)^k \cdot (ab) = a^k \cdot a \cdot b^k \cdot b = a^{k+1} \cdot b^{k+1}$$

Therefore, $P(K+1)$ is true whenever $P(K)$ is true.
Hence by principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example. 9 : Using the Principle of Mathematical Induction; prove that : $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$

for all $n \in N$.

Solution: Let $P(n)$ be defined as

$$P(n): \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$

Step I : $P(1)$: $\left(1-\frac{1}{2}\right) = \frac{1}{1+1}$ i.e. $\frac{1}{2} = \frac{1}{2}$, which is true

$\therefore$ $P(1)$ is true.

Step II : Let $P(n)$ be true for $n = k$, where $1 \le k < n$.

$$\therefore P(k): \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$$

Step III : Now we shall prove that $P(k+1)$ is true whenever $P(k)$ is true

$$P(k+1): \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) = \frac{1}{(k+1)} \times \left(1-\frac{1}{k+2}\right)$$

$$= \frac{1}{(k+1)}\left(\frac{k+2-1}{k+2}\right)$$

$$= \frac{1}{\cancel{k+1}} \times \frac{\cancel{k+1}}{k+2} = \frac{1}{k+2}$$

$\therefore$ $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 10: Use the Principle of Mathematical induction to prove that $n(n+1)(2n+1)$ is divisible by 6 for all $n \in N$.

Solution: Let $P(n)$ be defined as

$P(n)$: $n(n+1)(2n+1)$ is divisible by 6.

Step (1) : $P(1)$: $1(1+1)(2\times1+1) = 1\times2\times3 = 6$ which divisible by 6.

$\therefore$ $P(1)$ is true.

Step (2) : Let $P(n)$ be true for $n = k$ where $1 \le k < n$.

$\therefore$ $P(k)$: $k(k+1)(2k+1)$ is divisible by 6.

Step (3) : Now we shall prove that $P(k+1)$ is true whenever $P(k)$ is true.

$$P(k+1) : (k+1)[(k+1)+1][2(k+1)+1] = (k+1)(k+2)[(2k+1)+2]$$
$$= (k+1)(k+2)(2k+1) + 2(k+1)(k+2)$$
$$= (k+1)[k(2k+1)+2(2k+1)] + 2(k+1)(k+2)$$
$$= (k+1)k(2k+1) + 2(k+1)(2k+1) + 2(k+1)(k+2)$$
$$= k(k+1)(2k+1) + 2(k+1)[2k+1+k+2]$$
$$= k(k+1)(2k+1) + 2(k+1)(3k+3)$$
$$= k(k+1)(2k+1) + 6(k+1)^2$$

$k(k+1)(2k+1)$ is divisible by 6

$6(k+1)^2$ is also clearly divisible by 6.

$\therefore$ $P(k+1)$ is true whenever $P(k)$ is true.

Hence by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 11: By the Principle of Mathematical Induction, prove that for all $n \in N$, 3^{2n} when divided by 8, the remainder is 1 always.

Solution: Let $P(n)$ be defined as

$P(n)$: 3^{2n} when divided by 8, the remainder is 1.

Step (1): $P(1)$: $3^2 = 9 = 8+1$ which is true.

$\therefore P(1)$ is true.

Step (2): Let $P(n)$ be true for $n=k$, where $1 \le k < n$

$P(k)$: 3^{2k} when divided by 8, the remainder is 1.

$3^{2k} = 8m+1$ for some $m \in N$.

Step (3): We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

$$P(k+1) = 3^{2(k+1)} = 3^{2k}\cdot 3^2 = (8m+1)\times 9$$
$$= 72m+9$$
$$= 72m+8+1$$
$$= 8(9m+1)+1$$
$$= 8\lambda+1 \quad \text{when } \lambda = 9m+1 \in N.$$

Thus $P(k+1)$ is true whenever $P(k)$ is true.

Hence by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 12: Prove by Principle of Mathematical Induction that $(10^{2n-1}+1)$ is divisible by 11 for all $n \in N$.

Solution: Let $P(n)$ be defined as

$P(n)$: $(10^{2n-1}+1)$ is divisible by 11.

Step (1): $P(1)$: $10^{2\times1-1}+1 = 10+1 = 11$ which is divisible by 11

$\therefore P(1)$ is true.

Step (2): Let $P(n)$ be true $n=k$ where $1 \le k < n$

$\therefore P(k)$: $(10^{2k-1}+1)$ is divisible by 11.

$\Rightarrow (10^{2k-1}+1) = 11m$ for some $m \in N$.

Step (3): We shall now prove that $P(K+1)$ is true whenever $P(K)$ is true.

$P(K+1)$: $10^{2(K+1)-1}+1 = 10^{(2K+2-1)}+1$

$= 10^{2K-1+2}+1$

$= 10^{2K-1} \times 10^2 + 1$

$= (11m-1)\,10^2+1$ [Since $10^{2K-1}+1 = 11m$]

$= (11m-1)\times 100+1$

$= 1100m - 100 + 1$

$= 1100m - 99$

$= 11(100m-9)$ which is divisible by 11.

Hence $P(K+1)$ is true whenever $P(K)$ is true.

Hence by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in N$.