Chapter 9 : Sequences and Series (XI)

Introduction

When we say that a collection of objects is listed in a sequence, we usually mean the collection is ordered in such a way that it has an identified first member, second member, third member and so on.
For example, the amount of money deposited in a bank, over a number of years form a sequence.

(i) Sequence of odd natural numbers
(ii) Sequence of even natural numbers.
(iii) Sequence of prime numbers.

Sequence, following specific pattern are called progressions.

Sequences

In mathematics, we define a sequence in a non-empty set X to be a map $f : N \to X$. If $X = R$, we call the sequence f a real sequence and if $X = C$, we call f a complex sequence.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division.
In this process we get 3, 3.3, 3.33, 3.333, ... and so on.
These quotients also form a sequence.

The various numbers occurring in a sequence are called terms. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots a_n$ etc.
The nth term is the number at the nth position of the sequence and is denoted by a_n (also called general term).

A sequence containing finite number of terms is called finite sequence.

A sequence is called infinite, if it is not a finite

Fibonacci sequence

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8, ... has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$

$$a_3 = a_1 + a_2$$

$$a_n = a_{n-2} + a_{n-1}, \quad n > 2.$$

This sequence is called Fibonacci sequence.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3, \ldots, a_n$ in succession.

Thus, a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type $\{1, 2, 3, \ldots k\}$.

Series

Let $a_1, a_2, a_3, \ldots a_n$ be a given sequence. Then, the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called the series associated with the given sequence.

The series is finite or infinite. Series are often represented in sigma notation as $\sum_{k=1}^{n} a_k$.

Solved Examples

1) Write the first three terms in each of the following sequences defined by the following:
(i) $a_n = 2n+5$ (ii) $a_n = \frac{n-3}{4}$

Sol^n: (i) $a_n = 2n+5$

$a_1 = 2(1)+5 = 7$

$a_2 = 2(2)+5 = 9$

$a_3 = 2(3)+5 = 11.$

(ii) $a_n = \frac{n-3}{4}$ $a_1 = \frac{1-3}{4} = -\frac{1}{2}$ $a_2 = \frac{2-3}{4} = -\frac{1}{4}$ $a_3 = 0.$

2) What is the 20^{th} term of the sequence defined by $a_n = (n-1)(2-n)(3+n)$?

Sol^n: $n = 20$ $a_{20} = (20-1)(2-20)(3+20) = -7866.$

3) Let the sequence a_n be defined as follows:
$a_1 = 1$, $a_n = a_{n-1} + 2$ for $n \geq 2$.
Find first five terms of the sequence.

Sol^n: $a_1 = 1$, $a_2 = a_1 + 2 = 1+2 = 3$

$a_3 = a_2 + 2 = 3+2 = 5$

$a_4 = a_3 + 2 = 5+2 = 7$

$a_5 = a_4 + 2 = 7+2 = 9$

Hence, the first five terms of the sequence are 1, 3, 5, 7 and 9. The corresponding series is $1+3+5+7+9+\ldots$

(4) The Fibonacci sequence is defined by

$a_1 = 1 = a_2$, $a_n = a_{n-1} + a_{n-2}$ $(n > 2)$

Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$.

Sol^n: $n = 1$, $\frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = \frac{1}{1} = 1$ $[\because a_1 = a_2 = 1]$

$n = 2$, $\frac{a_{n+1}}{a_n} = \frac{a_3}{a_2}$

$a_n = a_{n-1} + a_{n-2}$, $a_3 = a_2 + a_1 = 1 + 1 = 2$

$\frac{a_3}{a_2} = \frac{2}{1} = 2$ $\quad a_4 = a_3 + a_2 = 2 + 1 = 3$

$\frac{a_4}{a_3} = \frac{3}{2}$ $\quad a_5 = a_4 + a_3 = 3 + 2 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

$\frac{a_5}{a_4} = \frac{5}{3}$ $\quad \frac{a_6}{a_5} = \frac{8}{5}$

(5) Consider the sequence defined by $a_n = an^2 + bn + c$.
If $a_2 = 3$, $a_4 = 13$ and $a_7 = 113$, show that $3a_n = 17n^2 - 87n + 115$.

Sol^n: $a_2 = a(2)^2 + b(2) + c \Rightarrow 4a + 2b + c = 3$

$a_4 = a(4)^2 + b(4) + c \Rightarrow 16a + 4b + c = 13$

$a_7 = a(7)^2 + b(7) + c \Rightarrow 49a + 7b + c = 113$

Solving these equations, we get $a = \frac{17}{3}$, $b = -29$, $c = \frac{115}{3}$

$\therefore a_n = \frac{17}{3} a^2 + (-29) b + \frac{115}{3}$

$3a_n = 17n^2 - 87n + 115$.

Arithmetic Progression (A.P.)

A sequence $a_1, a_2, a_3, \ldots a_n$ is called arithmetic sequence or arithmetic progression if $a_n - a_{n-1} = d$, where a_1 is called first term and the constant d is called the common difference of the A.P.

nth term (general term) of the A.P. $a_n = a + (n-1)d$.

Sum of n term $\quad S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[a + l] \quad [l - \text{last term}]$

Remember,

$a_n = S_n - S_{n-1}$

Properties of an A.P.

(i) If a constant is added to each term of an A.P., the resulting sequence is also in A.P.

(ii) If a constant is subtracted from each term of an A.P, the resulting sequence is also in A.P.

(iii) If each term is multiplied by a constant, then the resulting sequence is also in A.P.

(iv) If each term of an A.P. is divided by an non-zero constant then the resulting sequence is also in A.P.

a = the first term
l = the last term
d = common difference
n = the number of terms.
S_n = sum of n terms.

Solved Examples

1) In an A.P. if m^{th} term is n and the n^{th} term is m, where $m \neq n$, find the p^{th} term.

Soln: $a_m = a + (m-1)d = n$ — (1)

$a_n = a + (n-1)d = m$ — (2)

Solving (1) & (2), we get $(m-n)d = n-m$ or $d = -1$.

$a = n+m-1$, $d = -1$.

then $a_p = a + (p-1)d$

$= (n+m-1) + (p-1)(-1)$

$a_p = n+m-p$

(2) If the sum of n terms of an A.P is $nP + \frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference.

Soln: $S_n = nP + \frac{1}{2}n(n-1)Q$

$S_1 = a_1 = P + \frac{1}{2} \times 0 = P$

$S_2 = 2P + \frac{1}{2} \times 2\,(2-1)Q = 2P + Q$

$a_1 + a_2 = 2P + Q \Rightarrow P + a_2 = 2P + Q \Rightarrow a_2 = P + Q$

common difference $d = a_2 - a_1 = P + Q - P = Q$.

3) The sum of n terms of two arithmetic progressions are in ratio $(3n+15) : (7n+15)$. Find the ratio of 12^{th} term. [7:16]

4) The income of a person is Rs 3,00,000 in the first year and he receives an increase of Rs. 10,000 to his income per year for the next 19 year. Find the total amount, he received in 20 year.

Arithmetic Mean

Given two numbers a and b. We can insert a number A between them so that a, A, b is an A.P. Such a number A is called the arithmetic mean.

$$A - a = b - A \quad \text{i.e.} \quad A = \frac{a+b}{2}.$$

More generally, given any two number a and b, we can insert as many numbers as we like between them such that the resulting sequence is an A.P.

$$A_n = a + nd, \quad d = \frac{b-a}{k+1} \quad \therefore A_n = \frac{n(b-a)}{k+1} + a$$

Example, k - no. of arithmetic means.

1) Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Soln : let $3, A_1, A_2, A_3, A_4, A_5, A_6, 24$

Here $a = 3$ $b = 24$ $n = 8$

$a_n = a + (n-1)d$, $3 + (8-1)d = 24 \Rightarrow d = 3$

$A_n = a + \frac{n(b-a)}{k+1}$ $k = 6$.

$$A_1 = 3 + \frac{1(24-3)}{6+1} = 3 + \frac{21}{7} = 6$$

$$A_2 = 3 + \frac{2(24-3)}{6+1} = 3 + 2\times 3 = 9$$

$A_3 = 12$ $A_4 = 15$, $A_5 = 18$, $A_6 = 21$.

Geometric Progress (G.P.)

A series in which the ratio of any term to the preceding term is the same throughout is called the Geometric Progression, and the constant ratio is called the common ratio of the G.P.

For example, $3+6+12+24+\ldots$ is a G.P. because $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \cdots = 2$.

Standard form of G.P.:

It follows that, given the first term a and C.R. r, the G.P. can be rewritten as

$$a, ar, ar^2 \ldots ar^{n-1}$$

according as it is finite or infinite.

nth term of a G.P.

Let a be the first term and r the common ratio.

$$a_n = ar^{n-1}$$

Remember,

(i) If $\underline{a}$ is the first term and $\underline{r}$ is the common ratio of a finite G.P, having m terms, then nth term from the end is given by ar^{m-n}.

(ii) The nth term from the end of a G.P with last term l and common ratio r is $\frac{l}{r^{(n-1)}}$.

Sum to n terms of a G.P.

Let the first term of a G.P. be a and the common ratio be r
Let us denote by S_n the sum to first n terms of G.P.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

Case 1 : if $r = 1$ $\quad S_n = a + a + \dots + a$ (n term) $= na$

Case 2 : if $r \neq 1$ $\quad S_n = \dfrac{a(1-r^n)}{1-r}$ or $S_n = \dfrac{a(r^n - 1)}{r-1}$

Geometric Mean (G.M.)

The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$.
Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2, \dots G_K$ be k numbers between a and b, such that $a, G_1, G_2, G_3, \dots, G_K, b$.

$$r = \left(\frac{b}{a}\right)^{\frac{1}{K+1}} \quad [\because b \text{ being } (K+2)^{th} \text{ term}].$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{K+1}}$$

Relationship Between A.M and G.M.

Let A be A.M and G be G.M of two given no. a and b.

$$A = \frac{a+b}{2} \quad G = \sqrt{ab} \quad A - G = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

We obtain the relationship $A \geq G$.

Solved Examples

1) Find the 10th and nth terms of the G.P 5, 25, 125, ...

Soln: $a = 5$ $\quad r = \frac{25}{5} = 5$ $\quad a_n = ar^{n-1}$

$$a_{10} = 5(5)^{10-1} = 5(5)^9 = (5)^{10}$$

$$a_n = 5(5)^{n-1} = 5^n.$$

2) Which term of G.P. 2, 8, 32, ... up to n term is 131072?

Soln: $a = 2$ $\quad r = \frac{8}{2} = 4$ $\quad a_n = ar^{n-1}$

$$a_n = 2(4)^{n-1}$$

Now, $2(4)^{n-1} = 131072 \Rightarrow (4)^{n-1} = 65536$

$(4)^{n-1} = (4)^8$ $\quad \therefore n-1 = 8$ $\quad n = 9.$

Hence 9th term.

3) In a G.P. the 3rd term is 24 and 6th term is 192. Find the 10th term.

Soln: $a_3 = 24$ $\quad a_6 = 192$ $\quad a_3 = ar^2$ $\quad a_6 = ar^5$

$\frac{a_6}{a_3} = \frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{192}{24}$ $\quad r^3 = 8$ $\quad r = 2$ $\quad a = 6$

$$a_{10} = 6(2)^9 = 3072$$

4) Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+\ldots$

Sol^n: $a=1$ $r=\frac{2}{3}$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

$$S_5 = 3\left(1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}$$

5) How many terms of the G.P $3, \frac{3}{2}, \frac{3}{4}, \ldots$, are needed to give the sum $\frac{3069}{512}$?

Sol^n: $a=3$ $r=\frac{1}{2}$ $S_n = \frac{3069}{512}$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{3\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$$

$$6\left(1-\frac{1}{2^n}\right) = \frac{3069}{512}$$

$$1-\frac{1}{2^n} = \frac{3069}{512\times 6} = \frac{3069}{3072} \Rightarrow \frac{1}{2^n} = 1-\frac{3069}{3072} = \frac{1}{1024}$$

$$2^n = 1024 = 2^{10} \Rightarrow n=10.$$

6) The sum of first terms of a G.P. is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Sol^n: Let first three terms be $\frac{a}{r}, a, ar$, Then $\frac{a}{r}+a+ar=\frac{13}{12}$

$\frac{a}{\cancel{r}}\times a\times a\cancel{r} = -1 \Rightarrow a=-1$ $\quad -\frac{1}{r}+(-1)+(-1)r = \frac{13}{12}$

$12r^2+25r+12=0$ solving $r=-\frac{3}{4}$ or $-\frac{4}{3}$

Thus three terms are $\frac{4}{3}, -1, \frac{3}{4}$ or $\frac{3}{4}, -1, \frac{4}{3}$.

7) Find the sum of the sequence 7, 77, 777, 7777, .. to n terms.

Sol^n: $\frac{77}{7} = 11 \neq \frac{777}{77}$. This is not a G.P.

However, we can relate to a G.P. by writing the term as $S_n = 7 + 77 + 777 + 7777 + \cdots +$ to n term.

$$S_n = \frac{7}{9}[9 + 99 + 999 + 9999 + \cdots \text{ to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1) + (10^2-1) + (10^3-1) + \cdots n \text{ term}]$$

$$= \frac{7}{9}[(10 + 10^2 + 10^3 + \cdots +) - (1 + 1 + 1 + \cdots)]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{7}{9}\left[\frac{10(10^n-1)}{9} - n\right]$$

8) A person has 2 parents, 4 grandparents, 8 great grandparents and so on. Find the number of his ancestors during then ten generations preceding his own.

Sol^n: $a = 2$ $\quad r = 2$ $\quad n = 10$

$$S_n = \frac{a(r^n-1)}{r-1} \qquad S_{10} = 2(2^{10}-1) = 2046$$

Hence, the number of ancestors proceding the person is 2046.

9) Insert three numbers between 1 and 256 so that the resulting sequence is a GP.

Soln: let G_1, G_2 & G_3 be three number such that $1, G_1, G_2, G_3, 256$ are in GP.

$$G_n = a\left(\frac{b}{a}\right)^{\frac{n}{K+1}} \qquad K = 3 \qquad a = 1 \qquad b = 256$$

$$G_1 = 1\left(\frac{256}{1}\right)^{\frac{1}{3+1}} = (256)^{1/4} = \left[(4)^4\right]^{1/4} = 4$$

$$G_2 = 1\left(\frac{256}{1}\right)^{\frac{2}{4}} = (256)^{1/2} = 16$$

$$G_3 = 1\left(\frac{256}{1}\right)^{\frac{3}{4}} = \left[(4)^4\right]^{3/4} = 4^3 = 64.$$

Hence $G_1 = 4$, $G_2 = 16$, $G_3 = 64$

Alternative

$a_n = ar^{n-1}$ $\quad n=5$ term $\quad r^4 = 256$ $\quad r = \pm 4.$

$G_1 = ar$ $\quad G_2 = ar^2$ $\quad G_3 = ar^3$

We can insert $\pm 4, 16, \pm 64$.

10) If AM and GM of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Soln: AM, $\frac{a+b}{2} = 10$ $\quad a+b = 20$

GM $\sqrt{ab} = 8$ $\quad ab = 64$

Solving, $a=4$, $b=16$ or $a=16$, $b=4$

(11) The number of bacteria in a certain culture double every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?

Soln: It is a G.P $a = 30$ $r = 2$

$$a_n = ar^{n-1} = 30(2)^{n-1}$$
$$a_{n+1} = ar^n = 30(2)^n = (30)2^n$$

$$a_3 = ar^2 = 30(2)^2 = 120$$
$$a_5 = ar^4 = 30(2)^4 = 480.$$

12) What will Rs 500 amounts to in 10 years after its deposite in a bank which pays annual interest rate of 10% compounded annually?

Soln
$$\text{Amount} = P\left(1+\frac{1}{R}\right)^n = 500\left(1+\frac{1}{10}\right)^{10}$$
$$= 500(1.1)^{10}$$

13) The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for second, 4 grains for the third and so on, doubling the number of the grains for the subsequent squares. How many grains would have to be given to the inventor? (There are 64 squares in the chess board).

Soln: The number of grains that would be given to the inventor $= 1+2+4+8+16+\cdots+(64\text{th term})$

$$S_{64} = \frac{1(2^{64}-1)}{2-1} = 2^{64}-1 \qquad \because S_n = \frac{a(r^n-1)}{r-1} \text{ for } r>1.$$

14) Sum the following geometric series to infinity:

$(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \cdots \infty$

Sol^n: Given series, first term $a = \sqrt{2}+1$ common ration $r = \frac{1}{\sqrt{2}+1}$.

$$r = \frac{1}{\sqrt{2}+1} = \frac{1}{(\sqrt{2}+1)} \times \frac{\sqrt{2}-1}{(\sqrt{2}-1)} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

$$S_\infty = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}} = \frac{\sqrt{2}+1}{2(\sqrt{2}-1)} \times \frac{\sqrt{2}+1}{(\sqrt{2}+1)}$$

$$S_\infty = \frac{4+3\sqrt{2}}{2}.$$

15) Evaluate $9^{\frac{1}{3}} \cdot 9^{\frac{1}{9}} \cdot 9^{\frac{1}{27}} + \cdots + \infty$.

Sol^n: $9^{\frac{1}{3}} \cdot 9^{\frac{1}{9}} \cdot 9^{\frac{1}{27}+\cdots+\infty} = 9^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\infty\right)}$

$$= 9^{\frac{1/3}{1-1/3}} = 9^{\frac{1/3}{2/3}}$$

$$= 9^{\frac{1}{2}}$$

$$= 3.$$

16) The sum of an infinite geometric series is 3. A series which is formed by the squares of its term also have the sum equal to 3. Find the series.

Sol^n: let a be first term

r common ratio $(|r|<1)$

then, $\frac{a}{1-r} = 3$, also $\frac{a^2}{1-r^2} = 3$ (Given)

Squaring $\frac{a}{1-r} = 3$, $\frac{a^2}{(1-r)^2} = 9$

Dividing $\frac{a^2}{1-r^2} \div \frac{a^2}{(1-r)^2} = \frac{3}{9} \Rightarrow \frac{(1-r)^2}{1-r^2} = \frac{1}{3}$

$\frac{1-r}{1+r} = \frac{1}{3} \Rightarrow 3-3r = 1+r \Rightarrow 4r = 2 \quad \therefore r = \frac{1}{2}$

$\frac{a}{1-\frac{1}{2}} = 3 \Rightarrow a = \frac{3}{2}$

Hence, the series is $a + ar + ar^2 + \ldots \infty$

$= \frac{3}{2} + \frac{3}{2}\left(\frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2}\right)^2 + \ldots + \infty$

$= \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots + \infty.$

17) The first term of a G.P. exceeds the second term by 2 and the sum to infinity is 50, find the G.P.

Soln: let a - first term r - common ratio $|r| < 1$.

$a_1 - a_2 = 2 \Rightarrow a - ar = 2 \Rightarrow a(1-r) = 2$ —(1)

$\frac{1}{1-r} = \frac{a}{2}$

$S_\infty = 50 \Rightarrow \frac{a}{1-r} = 50$ —(2)

$a \times \frac{a}{2} = 50 \quad a^2 = 100 \quad a = 10$

$10(1-r) = 2 \Rightarrow 1-r = \frac{1}{5} \Rightarrow r = \frac{4}{5}$

The G.P. $a, ar, ar^2, \ldots \infty$

$\Rightarrow 10, 10 \times \frac{4}{5}, 10 \times \left(\frac{4}{5}\right)^2, \ldots$

$\Rightarrow 10, 8, \frac{32}{5}, \ldots$

18) Use a geometric series to express 0.555.... as a rational number.

Soln: $0.555 = 0.5 + 0.05 + 0.005 + \ldots$

$=$ Sum of an infinite G.P with

$a = \frac{5}{10}$ $r = \frac{1}{10}$

$S_\infty = \frac{a}{1-r} = \frac{5}{10} \div \left(1 - \frac{1}{10}\right) = \frac{5}{10} \div \frac{9}{10} = \frac{5}{\cancel{10}} \times \frac{\cancel{10}}{9} = \frac{5}{9}$

$\therefore 0.555 = \frac{5}{9}$.

19) The arithmetic mean between two numbers is 34 and their geometric mean is 16. Find the numbers.

Soln: Let the numbers be a and b.

$AM = 34$ $\therefore \frac{a+b}{2} = 34 \Rightarrow a+b = 68$.

$GM = 16$ $\sqrt{ab} = 16 \Rightarrow ab = 256$

$(a-b)^2 = (a+b)^2 - 4ab$

$= (68)^2 - 4 \times 256$

$= 4624 - 1024$

$(a-b)^2 = 3600$

$\therefore a - \cancel{b} = 60$

$a + \cancel{b} = 68$

$2a = 128$

$a = 64$ $b = 4$.

$\therefore$ Required numbers are 64 and 4.

20) If the arithmetic mean between a and b be equal to n times geometric mean; find the ratio a : b.

Soln: $A.M = \frac{a+b}{2}$ $G.M. = \sqrt{ab}$

$AM = n(G.M)$ $\therefore \frac{a+b}{2} = n\sqrt{ab}$

$$\frac{a+b}{2\sqrt{ab}} = \frac{n}{1}$$

Applying componendo and dividendo, we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{n+1}{n-1} \Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{n+1}{n-1}$$

Taking square roots, we get $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{n+1}}{\sqrt{n-1}}$

Again applying componendo and dividendo, we get

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n-1}} \Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{(\sqrt{n+1})+\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n-1}}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n-1}}$$

Squaring both sides, $\dfrac{a}{b} = \dfrac{n+1+n-1+2\sqrt{n+1}\cdot\sqrt{n-1}}{n+1+n-1-2\sqrt{n+1}\cdot\sqrt{n-1}}$

$$\frac{a}{b} = \frac{2n+2\sqrt{n^2-1}}{2n-2\sqrt{n^2-1}}$$

$$\therefore \frac{a}{b} = \frac{n+\sqrt{n^2-1}}{n-\sqrt{n^2-1}}$$

Sum to n Terms of Special Series

(i) $S_n = 1+2+3+4+\cdots+n$, $S_n = \dfrac{n(n+1)}{2}$

(ii) $S_n = 1^2+2^2+3^2+\cdots+n^2$, $S_n = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $S_n = 1^3+2^3+3^3+\cdots+n^3$, $S_n = \left[\dfrac{n(n+1)}{2}\right]^2$

Solved Examples:

1) Find the sum to n terms of the series : $5+11+19+29+41+\cdots$

Sol^n : $S_n = 5+11+19+29+\cdots+a_{n-1}+a_n$.

$S_n = \quad 5+11+19+ \quad +a_{n-2}+a_{n-1}+a_n$

On subtraction,

$0 = 5+[6+8+10+12+\cdots+(n-1)\text{ terms}] - a_n$

$a_n = 5+\dfrac{n-1}{2}[2\times 6+(n-2)2]$

$= 5+\dfrac{n-1}{2}[12+(n-2)2]$

$a_n = 5+(n-1)(n+4) = n^2+3n+1$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2+3k+1) = \sum_{k=1}^{n} k^2 + 3\sum_{n=1}^{n} k + n$

$= \dfrac{n(n+1)(2+1)}{6} + \dfrac{3n(n+1)}{2} + n$

$= \dfrac{n(n+2)(n+4)}{3}$.

(2) Find the sum to n terms of the series whose n^{th} term is $n(n+3)$.

Sol^n: $a_n = n(n+3)$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)(n+5)}{3}$$

Miscellaneous

3) If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p-q)$, $(q-r)$, $(r-s)$ are also in G.P.

4) If a, b, c are in G.P and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, prove that x, y, z are in A.P.

5) If a, b, c, d and p are different real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$, then show that a, b, c and d are in G.P.

6) If p, q, r are in G.P. and the equations, $px^2+2qx+r=0$ and $dx^2+2ex+f=0$ have common root, then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in A.P.