BINOMIAL THEOREM & MATHEMATICAL INDUCTION

BINOMIAL THEOREM

If a, b $\in$ R and n $\in$ N, then

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_n\, a^0 b^n$

REMARKS :

1. If the index of the binomial is n then the expansion contains n + 1 terms.
2. In each term, the sum of indices of a and b is always n.
3. Coefficients of the terms in binomial expansion equidistant from both the ends are equal.
4. $(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^{2-} ... + (-1)^n\, {}^nC_0 a^0 b^n$.

GENERAL TERM AND MIDDLE TERMS IN EXPANSION OF (A + B)^N

$t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

t_{r+1} is called a general term for all r $\in$ N and $0 \le r \le n$. Using this formula we can find any term of the expansion.

MIDDLE TERM (S) :

1. In $(a + b)^n$ if n is even then the number of terms in the expansion is odd. Therefore there is only one middle term and it is $\left(\frac{n+2}{2}\right)^{th}$ term.
2. In $(a + b)^n$, if n is odd then the number of terms in the expansion is even. Therefore there are two middle terms and those are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

BINOMIAL THEOREM FOR ANY INDEX

If n is negative integer then n! is not defined. We state binomial theorem in another form.

$$(a+b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + ... \frac{+n(n-1)...(n-r+1)}{r!} a^{n-r} b^r + ...$$

Here $t_{r+1} = \frac{(n-1)(n-2)...(n-r+1)}{r!} a^{n-r} b$

THEOREM :

If n is any real number, a = 1, b = x and |x| < 1 then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + ...$$

Here there are infinite number of terms in the expansion, The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)\,x}{r!}, r \ge 0$$

Note...

(i) Expansion is valid only when $-1 < x < 1$

(ii) ${}^nC_r$ can not be used because it is defined only for natural number, so ${}^nC_r$ will be written as $\frac{n(n-1).........(n-r+1)}{r!}$

(iii) As the series never terminates, the number of terms in the series is infinite.

(iv) General term of the series $(1 + x)^{-n} = T_{r+1} \rightarrow (-1)^r$ $\frac{1 \;+\; x}{1 \;-\; x}$ if | x | < |

(v) General term of the series $(1 - x)^{-n} \rightarrow T_{r+1}$ $= \frac{(\;\;+1)(\;\;+2)...(\;\;+\;\;-1)}{r!} x$

(vi) If first term is not 1, then make it unity in the following way. $(a+x)^n = a^n (1 + x/a)^n$ if $\left|\frac{x}{a}\right| < 1$

REMARKS:

1. If $|x| < 1$ and n is any real number, then

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

The general term is given by

$$t_{r+1} = \frac{(-1)^r n(n-1)(n-2)...(n-r+1)}{r!}x^r$$

2. If n is any real number and $|b| < |a|$, then

$$= (a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n$$

$$= a^n\left(1+\frac{b}{a}\right)^n$$

Note...

While expanding $(a + b)^n$ where n is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index, $|x| < 1$

1. $\frac{1}{1+x} = (1+x)^{-1}$

$= 1 - x + x^2 - x^3 + x^4 - x^5 + ...$

2. $\frac{1}{1-x} = (1+x)^{-1}$

$= 1 + x + x^2 + x^3 + x^4 + x^5 + ...$

3. $\frac{1}{(1+x)^2} = (1+x)^{-2}$

$= 1 - 2x + 3x^2 - 4x^3 + ...$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2}$

$= 1 + 2x + 3x^2 + 4x^3 + ...$

BINOMIAL COEFFICIENTS

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ in the expansion of $(a+b)^n$ are called the binomial coefficients and denoted by $C_0, C_1, C_2, ..., C_n$ respectively

Now

$(1 + x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + ... + {}^nC_nx^n$ ..... (i)

Put $x = 1$.

$(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore$ $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore$ ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

$\therefore$ $C_0 + C_1 + C_2 + ... + C_n = 2^n$

$\therefore$ The sum of all binomial coefficients is 2^n.

Put $x = -1$, in equation (i),

$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\,{}^nC_n$

$\therefore$ $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\,{}^nC_n$

$\therefore$ ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\,{}^nC_n = 0$

$\therefore$ ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$

$\therefore$ $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$

$C_0, C_2, C_4, ...$ are called as even coefficients

$C_1, C_3, C_5 ...$ are called as odd coefficients

Let $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = k$

Now $C_0 + C_1 + C_2 + C_3 + ... + C_n = 2^n$

$\therefore$ $(C_0 + C_2 + C_4 + ...) + (C_1 + C_3 + C_5 ...) = 2^n$

$\therefore$ $k + k = 2^n$

$2k = 2^n$

$\therefore$ $k = \frac{2^n}{2}$

$\therefore$ $k = 2^{n-1}$

$\therefore$ $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

$\therefore$ The sum of even coefficients = The sum of odd coefficients $= 2^{n-1}$

Properties of Binomial Coefficient

For the sake of convenience the coefficients ${}^nC_0, {}^nC_1, ..., {}^nC_r, ... {}^nC_n$ are usually denoted by $C_0, C_1, ..., C_r, ..., C_n$ respectively.

(i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(ii) $C_0 - C_1 + C_2 - .... + (-1)^n C_n = 0$

(iii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

(iv) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$

(v) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(vi) $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$

Some Important Results

(i) $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots\dots + C_nx^n$,
Putting x = 1 and – 1, we get
$C_0 + C_1 + C_2 + \dots + C_n = 2^n$ and
$C_0 - C_1 + C_2 - C_3 + \dots\dots\dots(-1)^n C_n = 0$

(ii) Differentiating $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, on both sides we have, $n(1+x)^{n-1}$
$= C_1 + 2C_2x + 3C_3x^2 + \dots\dots + nC_nx^{n-1}$ ....(1)
x = 1
$\Rightarrow \quad n2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots\dots + nC_n$
x = –1
$\Rightarrow \quad 0 = C_1 - 2C_2 + \dots\dots + (-1)^{n-1} nC_n$.
Differentiating (1) again and again we will have different results.

(iii) Integrating $(1+x)^n$, we have,

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots\dots\dots + \frac{C_nx^{n+1}}{n+1}$$

(where C is a constant)

Put x = 0, we get $C = -\frac{1}{(n+1)}$

Therefore

$$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots\dots\dots + \frac{C_nx^{n+1}}{n+1} \quad \dots (2)$$

Put x = 1 in (2) we get

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \dots\dots\dots + \frac{C_n}{n+1}$$

Put x = – 1 in (2) we get,

$$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots\dots\dots\dots$$

Illustration

Find the coefficient of x^4 in the expansion of $\frac{1+x}{1-x}$ if $|x| < 1$

Sol. $\frac{1+x}{1-x} = (1+x)(1-x)^{-1}$

$$= (1+x)\left[1 + \frac{(-1)}{1!}(-x) + \frac{(-1)(-1-1)}{2!}(-x)^2\right.$$

$$\left. + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 \dots\text{to } \infty\right]$$

$= (1+x)(1 + x + x^2 + x^3 + x^4 + \dots\dots \text{ to } \infty)$

$= [1 + x + x^2 + x^3 + x^4 + \dots\dots\dots \text{ to } \infty] + [x + x^2 + x^3 + x^4 + \dots\dots\dots \text{ to } \infty]$

$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots\dots \text{ to } \infty$

Hence coefficient of $x^4 = 2$

Illustration

Find the square root of 99 correct to 4 places of deicmal.

Sol. $(99)^{1/2} = (100-1)^{1/2}\left[100\left(1 - \frac{1}{100}\right)\right]^{\frac{1}{2}}$

$$= \left[100\left(1 - \frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$= (100)^{1/2}[1 - 0]^{1/2} = 10(1 - 01)^{1/2}$

$$10\left[1 + \frac{\frac{1}{2}}{1!}(-01) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-01)^2 + \dots\dots\dots\text{to } \infty\right]$$

$= 10\ [1 - 0.005 - 0.0000125 + \dots\dots\dots \text{ to } \infty]$
$= 10\,(.9949875) = 9.94987 = 9.9499$

Multinomial Expansion

In the expansion of $(x_1 + x_2 + \dots\dots + x_n)^m$ where $m, n \in N$ and $x_1, x_2, \dots\dots\dots, x_n$ are independent variables, we have

(i) Total number of terms $= {}^{m+n-1}C_{n-1}$

(ii) Coefficient of $x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots\dots\dots x_n^{r_n}$ (where $r_1 + r_2 + \dots\dots\dots + r_n = m, r_i \in N \cup \{0\}$ is $\frac{m!}{r_1!r_2!\dots\dots r_n!}$

(iii) Sum of all the coefficients is obtained by putting all the variables x_i equal to 1.

Illustration

Find the total number of terms in the expansion of $(1 + a + b)^{10}$ and coefficient of a^2b^3.

Sol. Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $a^2b^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$