Chapter 2 - Inverse Trigonometric Functions.

Introduction

If "$\sin\theta = x$" which means "sine of angle θ is x," we can say that "θ is the angle whose sine is x" and express it as $\theta = \sin^{-1}x$.

Since the problem of finding θ when x is given, is the inverse of finding x when θ is given, the symbol $\sin^{-1}x$ is often read as the sine inverse of x or arc sine of x. The symbols $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\csc^{-1}x$ are defined and read in an analogous manner.

These functions are known as the inverse circular function.

$$\sin^{-1}x \neq (\sin x)^{-1}$$

Illustration: To find the positive angles less than 2π

(i) $\theta = \sin^{-1}\frac{1}{\sqrt{2}}$ (ii) $\theta = \sec^{-1}2$ (iii) $\theta = \tan^{-1}(-1)$

(i) Since sine ratio is positive in 1st and 2nd quadrant.

Clearly $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

(ii) Secant is positive in 1st and 4th quadrant.

$\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$

(iii) Tan is positive in 1st and 3rd quadrant. And negative in 2nd and 4th quadrant.

$\therefore \theta = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$

Principal Values

	Function	Domain	Range (Principal value)
1.	$y=\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y=\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
3.	$y=\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y=\cot^{-1}x$	R	$[0, \pi]$
5.	$y=\sec^{-1}x$	$x \geq 1$ or $x \leq -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y=\text{cosec}^{-1}x$	$x \geq 1$ or $x \leq -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

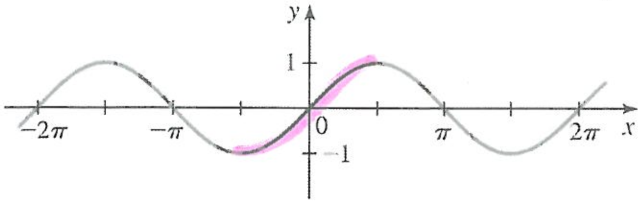
S.No.	Function	Domain	Range
(i)	$y=\sin^{-1}x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii)	$y=\cos^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii)	$y=\tan^{-1}x$	$x \in R$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
(iv)	$y=\cot^{-1}x$	$x \in R$	$0 < y < \pi$
(v)	$y=\text{cosec}^{-1}x$	$x \leq -1$ or $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$
(vi)	$y=\sec^{-1}x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi$; $y \neq \frac{\pi}{2}$

Principal values for $x \geq 0$	Principal values for $x < 0$
$0 \leq \sin^{-1}x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1}x < 0$
$0 \leq \cos^{-1}x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}x \leq \pi$
$0 \leq \tan^{-1}x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}x < 0$
$0 < \cot^{-1}x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1}x < \pi$
$0 \leq \sec^{-1}x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1}x \leq \pi$
$0 < \text{cosec}^{-1}x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \text{cosec}^{-1}x < 0$

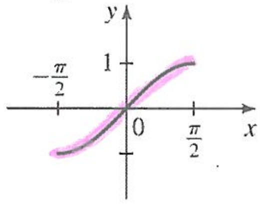
Inverse Trigonometric Functions and Their Graphs

DEFINITION: The **inverse sine function**, denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

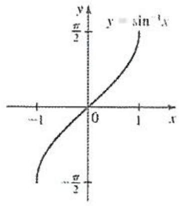
$$\sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



$y = \sin x$

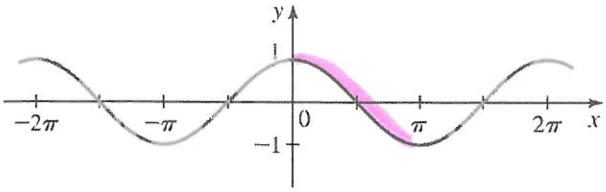


$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

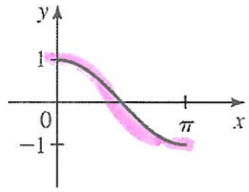


DEFINITION: The **inverse cosine function**, denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

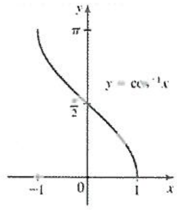
$$\cos x, \quad 0 \le x \le \pi$$



$y = \cos x$

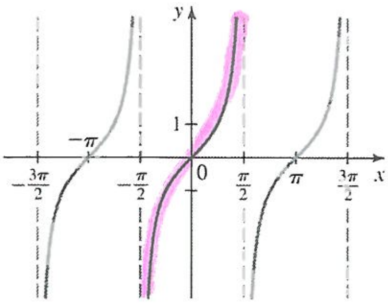


$y = \cos x,\ 0 \le x \le \pi$

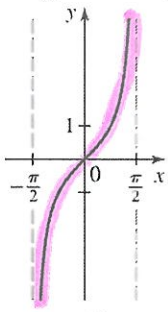


DEFINITION: The **inverse tangent function**, denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

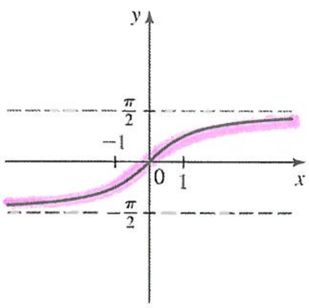
$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$y = \tan x$



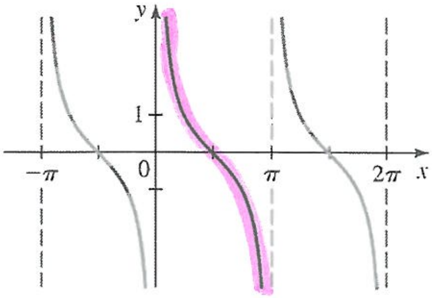
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



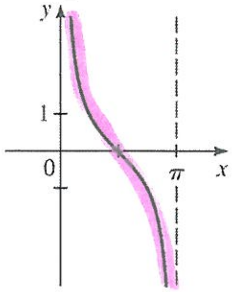
$y = \tan^{-1} x$

DEFINITION: The **inverse cotangent function**, denoted by $\cot^{-1} x$ (or $\operatorname{arccot} x$), is defined to be the inverse of the restricted cotangent function

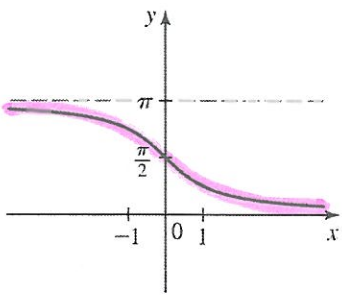
$$\cot x, \quad 0 < x < \pi$$



$y = \cot x$



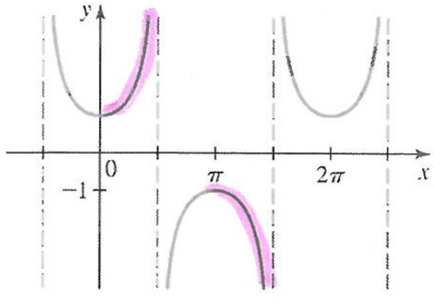
$y = \cot x,\ 0 < x < \pi$



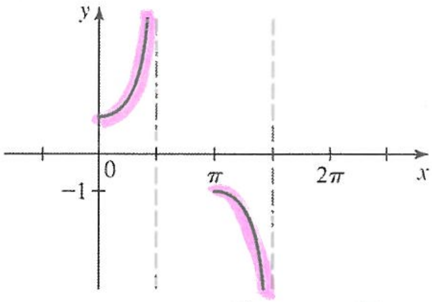
$y = \cot^{-1} x$

DEFINITION: The **inverse secant function**, denoted by $\sec^{-1} x$ (or $\operatorname{arcsec} x$), is defined to be the inverse of the restricted secant function

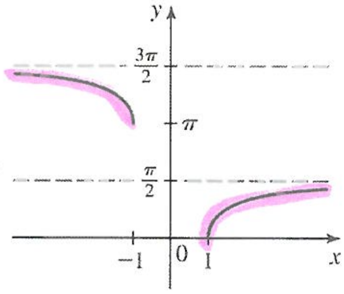
$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad \text{[or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks]}$$



$y = \sec x$



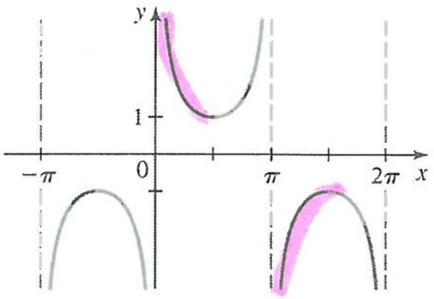
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$



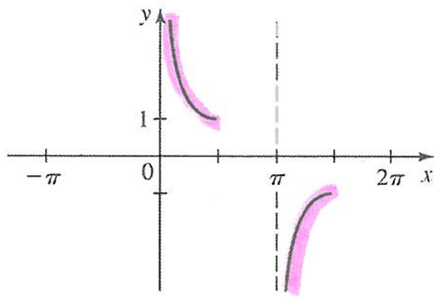
$y = \sec^{-1} x$

DEFINITION: The **inverse cosecant function**, denoted by $\csc^{-1} x$ (or $\operatorname{arccsc} x$), is defined to be the inverse of the restricted cosecant function

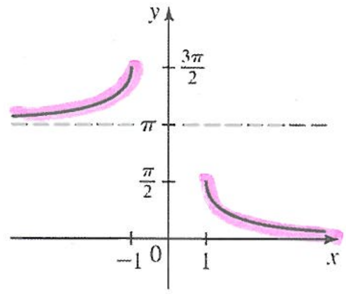
$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad \text{[or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks]}$$



$y = \csc x$



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$



$y = \csc^{-1} x$

IMPORTANT: Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

with

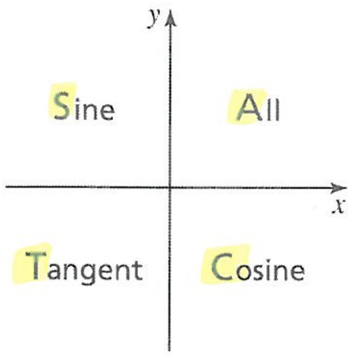
$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES:



(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(f) $\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

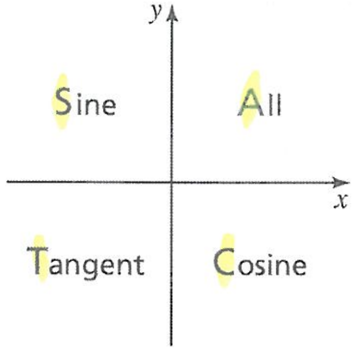
$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.



Solution: We have

$$\sec^{-1} 1 = 0, \qquad \sec^{-1}(-1) = \pi, \qquad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \qquad \sec \pi = -1, \qquad \sec \frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Note that $\sec \dfrac{2\pi}{3}$ is also -2, but

$$\sec^{-1}(-2) \neq \frac{2\pi}{3}$$

since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Find

$$\tan^{-1} 0 \qquad \cot^{-1} 0 \qquad \cot^{-1} 1 \qquad \sec^{-1} \sqrt{2} \qquad \csc^{-1} 2 \qquad \csc^{-1} \frac{2}{\sqrt{3}}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES: We have

$$\tan^{-1} 0 = 0, \quad \cot^{-1} 0 = \frac{\pi}{2}, \quad \cot^{-1} 1 = \frac{\pi}{4}, \quad \sec^{-1} \sqrt{2} = \frac{\pi}{4}, \quad \csc^{-1} 2 = \frac{\pi}{6}, \quad \csc^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{3}$$

EXAMPLES: Evaluate

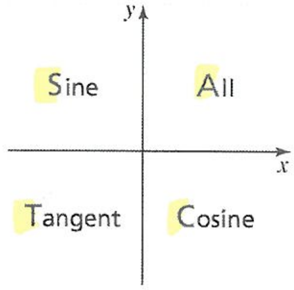
(a) $\sin\left(\arcsin \frac{\pi}{6}\right)$, $\arcsin\left(\sin \frac{\pi}{6}\right)$, and $\arcsin\left(\sin \frac{7\pi}{6}\right)$.

(b) $\sin\left(\arcsin \frac{\pi}{7}\right)$, $\arcsin\left(\sin \frac{\pi}{7}\right)$, and $\arcsin\left(\sin \frac{8\pi}{7}\right)$.

(c) $\cos\left(\arccos\left(-\frac{2}{5}\right)\right)$, $\arccos\left(\cos \frac{2\pi}{5}\right)$, and $\arccos\left(\cos \frac{9\pi}{5}\right)$.

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1, 1] \qquad \text{and} \qquad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$



Therefore

(a) $\sin\left(\arcsin \frac{\pi}{6}\right) = \arcsin\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$, but

$$\arcsin\left(\sin \frac{7\pi}{6}\right) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

or

$$\arcsin\left(\sin \frac{7\pi}{6}\right) = \arcsin\left(\sin\left(\pi + \frac{\pi}{6}\right)\right) = \arcsin\left(-\sin \frac{\pi}{6}\right) = -\arcsin\left(\sin \frac{\pi}{6}\right) = -\frac{\pi}{6}$$

(b) $\sin\left(\arcsin \frac{\pi}{7}\right) = \arcsin\left(\sin \frac{\pi}{7}\right) = \frac{\pi}{7}$, but

$$\arcsin\left(\sin \frac{8\pi}{7}\right) = \arcsin\left(\sin\left(\frac{\pi}{7} + \pi\right)\right) = \arcsin\left(-\sin \frac{\pi}{7}\right) = -\arcsin\left(\sin \frac{\pi}{7}\right) = -\frac{\pi}{7}$$

(c) Similarly, since $\arccos x$ is the inverse of the restricted cosine function, we have

$$\cos(\arccos x) = x \text{ if } x \in [-1, 1] \qquad \text{and} \qquad \arccos(\cos x) = x \text{ if } x \in [0, \pi]$$

Therefore $\cos\left(\arccos\left(-\frac{2}{5}\right)\right) = -\frac{2}{5}$ and $\arccos\left(\cos \frac{2\pi}{5}\right) = \frac{2\pi}{5}$, but

$$\arccos\left(\cos \frac{9\pi}{5}\right) = \arccos\left(\cos\left(2\pi - \frac{\pi}{5}\right)\right) = \arccos\left(\cos \frac{\pi}{5}\right) = \frac{\pi}{5}$$

Solved Examples

1) Find the principal value of the following :

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iii) $\operatorname{cosec}^{-1}(2)$ (iv) $\tan^{-1}(-\sqrt{3})$.

Soln: (i) Let $\sin^{-1}\left(\frac{1}{2}\right) = \theta$, then $\sin\theta = \frac{1}{2}$

Range of principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\theta = \sin\frac{\pi}{6}$ $\therefore \theta = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Hence, principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

(ii) Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$, then $\sin\theta = -\frac{\sqrt{3}}{2}$

Range (Principal value) of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\theta = -\sin\frac{\pi}{3} = \sin\left(-\frac{\pi}{3}\right)$ $\therefore \theta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) Let $\operatorname{cosec}^{-1}(2) = \theta$, then $\operatorname{cosec}\theta = 2$

Range (Principal value) of $\operatorname{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\operatorname{cosec}\theta = \operatorname{cosec}\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

(iv) Let $\tan^{-1}(-\sqrt{3}) = \theta$, then $\tan\theta = -\sqrt{3}$

Range (Principal Value) of $\tan^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan\theta = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right) \Rightarrow \theta = -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(3) (i) Write the principal value of $[\tan^{-1}(-\sqrt{3})+\tan^{-1}(1)]$

(ii) Evaluate $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.

Sol^n : (i) $\tan^{-1}(-\sqrt{3})+\tan^{-1}(1) = -\tan^{-1}(\sqrt{3}) + \frac{\pi}{4}$

$$= -\frac{\pi}{3}+\frac{\pi}{4} = -\frac{\pi}{12}.$$

$$\frac{(-4+3)\pi}{12}$$

(ii) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{3\pi}{5} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ which is the principal value branch of $\sin^{-1}x$.

Now, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left[\sin\left(\pi-\frac{3\pi}{5}\right)\right] = \sin^{-1}\left(\sin\frac{2\pi}{5}\right)$

$$= \frac{2\pi}{5} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

$$\therefore \sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}.$$

4) (i) Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

(ii) Write the principal value of $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$

Sol^n: (i) Range (Principal value) of $\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} \notin \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Now $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right] = \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$ $[\because \tan^{-1}(-\theta) = -\tan^{-1}\theta]$

(ii) $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}\left[-\sin\frac{\pi}{2}\right] = \tan^{-1}(-1) = -\tan^{-1}(1)$

$= -\frac{\pi}{4}$ $[\because \sin(-\theta) = -\sin\theta]$

(5) Find the principal value of $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$.

Solⁿ: $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\therefore \sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\} = \sin^{-1}\left\{\cos\left(\frac{\pi}{3}\right)\right\} = \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

6) Find the of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solⁿ: $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{13\pi}{6} \notin [0,\pi]$

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right] = \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$= \frac{\pi}{6} \in [0,\pi]$

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$.

7) Find the value of $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$

Solⁿ: Let $\cos^{-1}\frac{7}{25} = x$ $\quad \cos x = \frac{7}{25}$

$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(\frac{7}{25}\right)^2} = \frac{24}{25}$

$\cot x = \frac{\cos x}{\sin x} = \frac{7}{25} \div \frac{24}{25} = \frac{7}{24}$

$\Rightarrow x = \cot^{-1}\left(\frac{7}{24}\right) = \cos^{-1}\left(\frac{7}{25}\right)$

$\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] = \frac{7}{24}$.

Properties of Inverse Trigonometric function

1) Briefly stated, an inverse function of a trigonometric function is the function $f^{-1}(x)$ such that

$$f[f^{-1}(x)] = x$$

$$f^{-1}(f(x)) = x.$$

For example, $\sin^{-1}(\sin x) = x$ for $\forall\ x \in [-1,1]$

$$\sin^{-1}(\sin\theta) = \theta \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

(2) **Theorem:** (i) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$, $x \in R - (-1,1)$.

(ii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$, $x \in R - (-1,1)$

(iii) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$, $x > 0$ and $\pi + \cot^{-1}x$, $x < 0$.

(3) **Theorem:** (i) $\sin^{-1}(-x) = -\sin^{-1}x$ (iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$.

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$

(4) **Theorem:** (i) $\cos^{-1}(-x) = \pi - \cos^{-1}x$ (ii) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$.

(5) **Theorem:** (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

(6) (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, if $|x|<1$

(7) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if $|x|<1$.

(8) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

9) (i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(ii) $\operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

(iii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

(10) (i) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(ii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{(1-x^2)(1-y^2)}\right\}$

Solved Examples

1) Prove that : $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x}$

$$= \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x} = \sec^{-1}\sqrt{1+x^2}$$

Solⁿ: Let $\tan^{-1}x = \theta$, then $x = \tan\theta$

$$\sin^{-1}\frac{x}{\sqrt{1+x^2}} = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right) = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$$

$$= \sin^{-1}\left(\frac{\sin\theta}{\cos\theta} \div \frac{1}{\cos\theta}\right) = \sin^{-1}\left(\frac{\sin\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta}\right)$$

$$= \sin^{-1}(\sin\theta)$$

$$= \theta$$

$$\sin^{-1}\frac{x}{\sqrt{1+x^2}} = \tan^{-1}x$$

$$\cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) = \cos^{-1}\left(\frac{1}{\sec\theta}\right) = \cos^{-1}(\cos\theta) = \theta$$

(2) Write the following functions in simplest form :

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$ (ii) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ (iii) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Solⁿ: (i) Put $x = a\sin\theta$ $\theta = \sin^{-1}\frac{x}{a}$

$$\therefore \frac{x}{\sqrt{a^2-x^2}} = \frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \frac{\cancel{a}\sin\theta}{\cancel{a}\cos\theta} = \tan\theta$$

$$\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

(ii) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

We know that $-2\sin A\cdot\sin B = \cos(A+B) - \cos(A-B)$

put $A = B$

then, $-2\sin A\cdot\sin A = \cos(A+A) - \cos(A-A)$

$-2\sin^2 A = \cos 2A - \cos 0^\circ$

$\therefore\ 2\sin^2 A = 1 - \cos 2A \qquad [\cos 0^\circ = 1]$

Similarly, $2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$

put $A = B$, $\quad 2\cos^2 A = 1 + \cos 2A$

$\therefore\ 1 - \cos x = 2\sin^2\dfrac{x}{2}, \quad 1 + \cos x = 2\cos^2\dfrac{x}{2}$

$$\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\frac{x}{2}}{\cancel{2}\cos^2\frac{x}{2}}} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

$$\therefore\ \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{x}{2}$$

(iii) $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$ [Dividing num and deno by $\cos x$]

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$$

$$= \frac{\pi}{4} - x \qquad \left[\because \tan(45+\theta) = \frac{1+\tan}{1-\tan\theta}\right.$$

$$\left.\&\ \tan(45-\theta) = \frac{1-\tan\theta}{1+\tan\theta}\right.$$

3) Write the following in the simplest form :

(i) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ (ii) $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$

(iii) $\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$ (iv) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{1+x^2-\sqrt{1-x^2}}\right)$

Sol^n: (i) put $x = \tan\theta$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\cancel{\cos\theta}}\times\frac{\cancel{\cos\theta}}{\sin\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$1-\cos\theta = 2\sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

ii) $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$ put $x = \sec\theta$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = \tan\theta$$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta = \frac{\pi}{2}-\sec^{-1}x$$

$$= \operatorname{cosec}^{-1}x \qquad \left[\because \sec^{-1}x+\operatorname{cosec}^{-1}x = \frac{\pi}{2}\right].$$

(iii) $\sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$

put $x = \sin\theta$ $\quad \sqrt{1-x^2} = \cos\theta$

$\sqrt{x} = \sin\alpha$ $\quad \sqrt{1-x} = \cos\alpha$

$\therefore \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\left(\sqrt{1-x^2}\right)\right) = \sin^{-1}\left(\sin\theta \cdot \cos\alpha - \sin\alpha\cos\theta\right)$

$\left[\because \sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B\right]$

$$= \sin^{-1}\left[\sin(\theta-\alpha)\right] = \theta - \alpha$$

$$= \sin^{-1}x - \sin^{-1}\sqrt{x}.$$

(iv) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2 = \cos 2\theta$ $\qquad \theta = \frac{1}{2}\cos^{-1}(x^2)$

$$\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}$$

$\left(1+\cos 2\theta = 2\cos^2\theta \qquad 1-\cos 2\theta = 2\sin^2\theta\right)$

$$= \frac{\sqrt{2}\cos 0 + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

Dividing numerator and denominator by $\cos\theta$

$$= \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)$$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right) = \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2).$$

4) Prove that (i) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$ $x \in [\frac{-1}{2}, 1]$

(ii) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$ $x \in [-\frac{1}{2}, \frac{1}{2}]$

Solⁿ: (i) put $\cos^{-1}x = \theta$, $x = \cos\theta$

$4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$

$\therefore \cos^{-1}(4x^3-3x) = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x$.

(ii) put $\sin^{-1}x = \theta$ $x = \sin\theta$

$3x - 4x^3 = 3\sin\theta - 4\sin^3\theta = \sin 3\theta$

$\therefore \sin^{-1}(3x-4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1}x$.

5) Write the value of $\cot^{-1}(\sqrt{1+x^2} - x)$ in the simplified form.

Solⁿ: put $x = \cot\theta$

$\sqrt{1+x^2} - x = \sqrt{1+\cot^2\theta} - \cot\theta = \operatorname{cosec}\theta - \cot\theta$

$= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} = \frac{\cancel{2}\cdot\sin^{\cancel{2}}\frac{\theta}{2}}{\cancel{2}\sin\cancel{\frac{\theta}{2}}\cdot\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

$\therefore \cot^{-1}(\sqrt{1+x^2} - x) = \cot^{-1}(\tan\frac{\theta}{2}) = \cot^{-1}[\cot(\frac{\pi}{2} - \frac{\theta}{2})]$

$= \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x$

6) Write $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a>0$, $-\frac{a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$ in the simplest form.

Soln: put $x = a\tan\theta$ $\quad\theta = \tan^{-1}\frac{x}{a}$

$$\frac{3a^2x - x^3}{a^3 - 3ax^2} = \frac{3a^2\, a\tan\theta - a^3\tan^3\theta}{a^3 - 3a\cdot a^2\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \tan 3\theta$$

$$\therefore \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\frac{x}{a}$$

7) Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos\left(\frac{1-x}{1+x}\right)$

Soln: Since $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\therefore 2\tan^{-1}\sqrt{x} = \cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$\therefore \tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right).$$

8) Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$.

Soln: Since $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$$\left(\frac{x}{y} - \frac{x-y}{x+y}\right) \div 1 + \frac{x}{y}\times\frac{x-y}{x+y} = \frac{x(x+y) - y(x-y)}{\cancel{y(x+y)}} \div \frac{y(x+y) + x(x-y)}{\cancel{y(x+y)}}$$

$$= \frac{x^2 + \cancel{xy} - \cancel{yx} + y^2}{\cancel{yx} + y^2 + x^2 - \cancel{yx}} = \frac{x^2+y^2}{x^2+y^2} = 1$$

$$\therefore \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

9) Prove that (i) $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}$, $x^2 < \frac{1}{3}$

(ii) $\sin\left(\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right) = 1.$

Solⁿ: (i) $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\therefore \tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left[\left(x+\frac{2x}{1-x^2}\right) \div \left(1 - x \times \frac{2x}{1-x^2}\right)\right]$

$= \tan^{-1}\left[\left(\frac{x(1-x^2)+2x}{1-x^2}\right) \div \left(\frac{(1-x^2)-2x^2}{1-x^2}\right)\right]$

$= \tan^{-1}\left[\frac{x-x^3+2x}{1-3x^2}\right] = \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = RHS$

(ii) put $x = \tan\theta$

$\frac{1-\tan^2\theta}{2\tan\theta} = \cot 2\theta \qquad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$LHS = \sin\left[\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$= \sin\left[\tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta)\right]$

$= \sin\left[\tan^{-1}\left(\tan\left(\frac{\pi}{2}-2\theta\right)\right) + 2\theta\right]$

$= \sin\left(\frac{\pi}{2} - \cancel{2\theta} + \cancel{2\theta}\right)$

$= \sin\frac{\pi}{2}$

$= 1$

$= RHS.$

(10) Prove that : (i) $2\left[\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right]=\pi$

(ii) $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

(iii) $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15$.

Soln: (i) $LHS = 2\left[\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right]$

$$= 2\left[\tan^{-1}1+\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right] = 2\left[\tan^{-1}1+\tan^{-1}1\right] = 2\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= 2\times\frac{\pi}{2} = \pi = RHS.$$

(ii) $LHS = \cot\left(\frac{\pi}{4}-2\cot^{-1}3\right) = \cot\left(\frac{\pi}{4}-2\tan^{-1}\frac{1}{3}\right)$

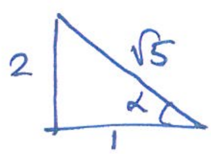
$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$ $\therefore$ $= \cot\left(\frac{\pi}{4}-\tan^{-1}\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right)$

$$= \cot\left(\frac{\pi}{4}-\tan^{-1}\frac{2\times9}{3\times8}\right) = \cot\left(\frac{\pi}{4}-\tan^{-1}\frac{3}{4}\right)$$

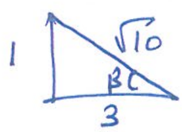
$$= \cot\left(\tan^{-1}1-\tan^{-1}\frac{3}{4}\right) = \cot\left[\tan^{-1}\frac{1-3/4}{1+3/4}\right]$$

$$= \cot\left(\tan^{-1}\frac{1}{7}\right) = \cot(\cot^{-1}7) = 7 = RHS.$$

(iii) Let $\tan^{-1}2=\alpha$ $\therefore \tan^{-1}2 = \sec^{-1}\sqrt{5}$



Let $\cot^{-1}3=\beta$ $\therefore \cot^{-1}3 = \operatorname{cosec}^{-1}\sqrt{10}$



$\therefore$ $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3) = \sec^2\alpha+\operatorname{cosec}^2\beta$

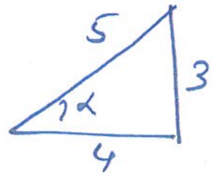
$= 5+10$

$= 15$

$= RHS.$

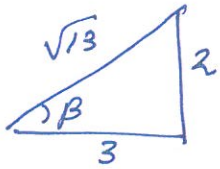
11) Prove the following: $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$.

Soln: Let $\sin^{-1}\frac{3}{5} = \alpha$, $\sin\alpha = \frac{3}{5}$



$\therefore \cos\alpha = \frac{4}{5}$

Let $\cot^{-1}\frac{3}{2} = \beta$ $\cot\beta = \frac{3}{2}$



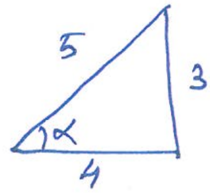
$\sin\beta = \frac{2}{\sqrt{13}}$, $\cos\beta = \frac{3}{\sqrt{13}}$

$$\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \cos(\alpha+\beta)$$
$$= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$= \frac{4}{5} \times \frac{3}{\sqrt{13}} - \frac{3}{5} \times \frac{2}{\sqrt{13}}$$
$$= \frac{12-6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}}$$

$$\therefore \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}.$$

12) Find the value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$.

Soln: Let $\cos^{-1}\frac{4}{5} = \alpha$, $\tan\alpha = \frac{3}{4}. \Rightarrow \alpha = \tan^{-1}\frac{3}{4}$



$\therefore \cos^{-1}\frac{4}{5} = \alpha = \tan^{-1}\frac{3}{4}$

Hence, $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right] = \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right]$

$$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right] = \tan\left[\tan^{-1}\frac{17}{6}\right]$$

$$= \frac{17}{6}.$$

(13) Prove that : $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

[Hint... $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)(1-y^2)}\}$]

(14) Prove that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$

[Hint... $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$.]

(15) Prove that : (i) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

(ii) $\tan^{-1}5 - \tan^{-1}3 + \tan^{-1}\frac{7}{9} = \frac{\pi}{4}$

(iii) $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{4}{5}$

(16) Prove that : (i) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$

(ii) $\tan^{-1}\frac{1-x}{1+x} - \tan^{-1}\frac{1-y}{1+y} = \sin^{-1}\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$

(17) Prove that (i) $\tan^{-1}n + \cot^{-1}(n+1) = \tan^{-1}(n^2+n+1)$

(ii) $\tan^{-1}\left(\frac{n}{n+1}\right) - \tan^{-1}(2n+1) = \frac{3\pi}{4}$

18) Prove that $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{5}{\sqrt{34}} = \tan^{-1}\left(\frac{27}{11}\right)$

19) Solve the following equations :

(i) $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$ [Ans. $x = \frac{\pi}{4}$]

(ii) $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x \quad (x > 0)$ [Ans. $x = \frac{1}{\sqrt{3}}$].