Mathematics (XI)

Chapter 6 : Linear Inequalities (XI)

Introduction

We have studied equations in one and two variables in earlier classes. We solved word problems by translating them into equations.

'Is it always possible to translate a statement problem in the form of an equation? for example, the height of all the students in your class is less than 160 cm.

We get certain statements involving a sign '$<$' (less than), '$>$' (greater than), '$\leq$' (less than or equal) and '$\geq$' (greater than or equal) which are known as inequalities.

The study of inequalities is very much used in the fields of Science, Mathematics, Statistics, Engineering, Economics etc.

INTERVALS

(a) Closed Interval : Let a and b be two real numbers, where $a < b$. Then the set of all real numbers x such that $a \leq x \leq b$ is called closed interval, which is denoted by $[a, b]$.

Thus $[a, b] = \{x \in R : a \leq x \leq b\}$

Here end points a and b are included, which are shown by closed circles.

(b) Open Interval : Set of all real numbers x such that $a < x < b$ is called open interval, which is denoted by (a, b)

Thus $(a, b) = \{x \in R : a < x < b\}$

End points a and b are excluded, which are shown by open circles.

(c) Semi-closed / Semi-open Intervals: Set of all real numbers x such that $a < x \leq b$ or $a \leq x < b$ are called semi-closed / semi-open intervals.

Thus $(a, b] = \{x \in R : a < x \leq b\}$ and $[a, b) = \{x \in R : a \leq x < b\}$

INEQUATIONS

A statement which involves variable and sign of inequality viz. $>, <, \geq$ or $\leq$ is an inequation.

(i) $ax+b < 0$, $ax+b \geq 0$ $(a \neq 0)$ are inequation in one variable
(ii) $ax+by < c$, $ax+by \geq c$ $(a \neq 0, b \neq 0)$ inequation in two variables.
(iii) $ax^2+bx+c < 0$ etc $(a \neq 0)$ are quadratic inequation in one variable.

Example:

(a) Ravi goes to market with Rs 200 to buy rice. The price of 1 kg packet rice is Rs. 30. If x denotes the number of packets of rice, then total amount spent by him is Rs. $30x$. Since he may not spent entire Rs. 200, Hence $30x < 200$.

(ii) Reshma has Rs. 120 and want to buy some register costing Rs 40 each and a pens costing Rs. 20 each. If x denotes number of register and y denotes no. of pen then total amount spent by her is $40x+20y \leq 120$.

Solutions of Inequations.

A solution of an inequation is the value of the variable(s), which makes it a true statement.

Consider the inequation $13x < 200$, where 'x' is non-negative.

when $x = 0$; LHS $= 0 < 200$ (RHS) which is true

$x = 1$; LHS $= 13 < 200$ (RHS) which is true

$x = 15$; LHS $= 195 < 200$ (RHS) which is true

$x = 16$; LHS $= 208 < 200$ (RHS) which is not true.

Hence, $13x < 200$ is not true for all $x > 16$.

Given statement is true for $x = 0, 1, 2, 3, \ldots\ldots 15$

General Rules :

(1) We can add or subtract the same number to (or from) both sides of an inequation.

Thus $a \leq b \Rightarrow a \pm x \leq b \pm x$

Any number can be transposed from one side to the other side with the sign of the reversed number reversed.

(2) We can multiply (or divide) both sides of an inequation by a positive number. Thus $a \leq b \Rightarrow 3a \leq 3b$.

If we multiply both sides of an inequation by a negative number, the signs of inequality '<' and '>' are reversed. $a \leq b \Rightarrow -2a \geq -2b$.

(3) If $\frac{a}{b} \geq \frac{c}{d}$ then; $ad \geq bc$ if same sign.

(4) We cannot take the square roots of an inequation.

$x^2 \leq 9 \not\Rightarrow x \leq +3$.

Rather $x^2 \leq a^2 \Rightarrow |x| \leq a \Rightarrow -a \leq x \leq a$.

Examples:

Example 1: Solve $30x < 200$ when (i) x is a natural number
(ii) x is an integer.

Sol^n: Given $30x < 200$

$$x < \frac{20\cancel{0}}{3\cancel{0}} \qquad x < \frac{20}{3}$$

(i) When x is natural number $x = 1, 2, 3, 4, 5, 6$
The solution set of inequality is $\{1, 2, 3, 4, 5, 6\}$

(ii) When x is an integer solution set is
$\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

Example 4: Solve $5x - 3 < 3x + 1$ when (i) x is an integer
(ii) x is a real number

Sol^n: We have $5x - 3 < 3x + 1$

$$2x < 4$$
$$x < 2$$

(i) When x is an integer $\{\ldots -4, -3, -2, -1, 0, 1\}$

(ii) When x is a real number then $x \in (-\infty, 2)$

Example 3: Solve $4x + 3 < 6x + 7$

Sol^n: $4x + 3 < 6x + 7 \Rightarrow 4x - 6x < 7 - 3$

$$-2x < 4$$
$$x > -2$$

Solution set $x \in (-2, \infty)$

Example 4: Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$

Sol^n: $\frac{5-2x}{\cancel{3}} \times \cancel{18}^{6} \leq \frac{x}{\cancel{6}} \times \cancel{18}^{3} - 5 \times 18$

$\cancel{30}_{10} - \cancel{12}_{4}x \leq \cancel{3}x - 5 \times 6 \times \cancel{3}$

$10 - 4x \leq x - 30$ Sol^n set $x \in [8, \infty)$

$-5x \leq -40$

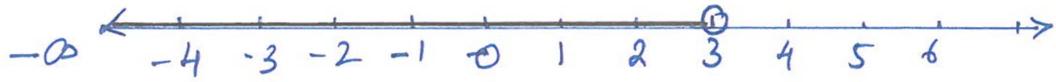
$x \geq 8$

Example 5: Solve $7x+3 < 5x+9$. Show the graph of the solution on number line.

Sol^n: Given $7x+3 < 5x+9$

$$2x < 6$$

$$x < 3 \qquad \text{sol}^n \text{ set } \; x \in (-\infty, 3)$$



Example 6: Solve $\frac{3x-4}{2} > \frac{x+1}{4} - 1$. Show graphically.

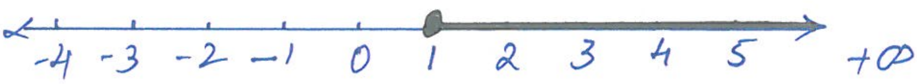
Sol^n: $\frac{(3x-4)}{\cancel{2}} \times \cancel{4}^{2} \geq \frac{(x+1)}{\cancel{4}} \times \cancel{4} - 1 \times 4$

$$6x - 8 \geq x + 1 - 4$$

$$6x - 8 \geq x - 3$$

$$5x \geq 5$$

$$x \geq 1 \qquad \text{Sol}^n \text{ set } \; x \in [1, \infty)$$



Example 7: The marks obtained by a student of class XI in first and second terminal examination are 62 and 48, respectively. Find the number of minimum marks he should get in the annual examination to have an average of at least 60 marks.

Soln: Let x be the marks obtained in annual exams

then $\frac{x + 62 + 48}{3} \geq 60$

$$x + 110 \geq 180$$

$$x \geq 70$$

This student should obtain minimum 70 marks.

Example 8: Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Sol: Let x be smaller of two consecutive number, then other number is $x+2$

$x > 10$ ...(1) and $x + (x+2) < 40$

$2x + 2 < 40$

$x < 19$ —— (2)

from (1) & (2) $10 < x < 19$.

Since x is odd, $\therefore$ $x = \{11, 13, 15, 17\}$

pairs $(11, 13)$, $(13, 15)$, $(15, 17)$, $(17, 19)$

Example 9: Solve the system of inequation.

$x - 2 > 0$, $3x < 18$

Soln: $x - 2 > 0 \Rightarrow x > 2$ —— (1)

$3x < 18 \Rightarrow x < 6$ —— (2)

From (1) & (2) $x \in (2, 6)$

Example 10: Solve the following system of inequation.

$2x - 7 > 5 - x$, $11 - 5x \leq 1$

Soln: $2x - 7 > 5 - x$, $3x > 12$ $x > 4$ —— (1)

$11 - 5x \leq 1$, $-5x \leq -10$, $x \geq 2$

Soln set $x \in (4, \infty)$

Graphical Solutions of Linear Inequalities in Two Variables.

If a, b, c are real numbers, then:
$ax+by<c$, $ax+by\le c$, $ax+by>c$, $ax+by\ge c$
are called linear inequations in two variables x and y.
We know that the graph of $ax+by=c$ is a straight line, which divides the xy-plane into two parts.
$ax+by\le c$ and $ax+by\ge c$ are called as closed half-spaces.
$ax+by<c$ and $ax+by>c$ are called as open half-spaces.

Approach:

(i) Convert the given inequation; say $ax+by\le c$ into the equation $ax+by=c$, which is a straight line in xy-plane.

(ii) Find the points where $ax+by=c$ meets the x-axis $(y=0)$ and y-axis $(x=0)$.

(iii) Join the two points obtained in step (ii) and obtain the graph of straight line $ax+by=c$.

(iv) (I) If the inequality is $ax+by<c$, draw the dotted line.
(II) If the inequality is $ax+by\le c$ draw the thick line.

(v) Select the point (0,0), if it does not lie on the line. Substitute the co-ordinates of the points in the given inequation. If the inequation is satisfied, shade the portion of the plane in which the point (0,0) lies, otherwise shade the portion of the plane in which the point (0,0) does not lie.

Note: If the inequalities are $ax+by\le c$ and $ax+by\ge c$, the points on the line are in the shaded region while in the case of inequalities $ax+by<c$ and $ax+by>c$, the points on the line are not in shaded region.

Example 1 : Solve $3x+2y>6$ graphically.

Let $3x+2y=6$

$x=0$; $2y=6$, $y=3$

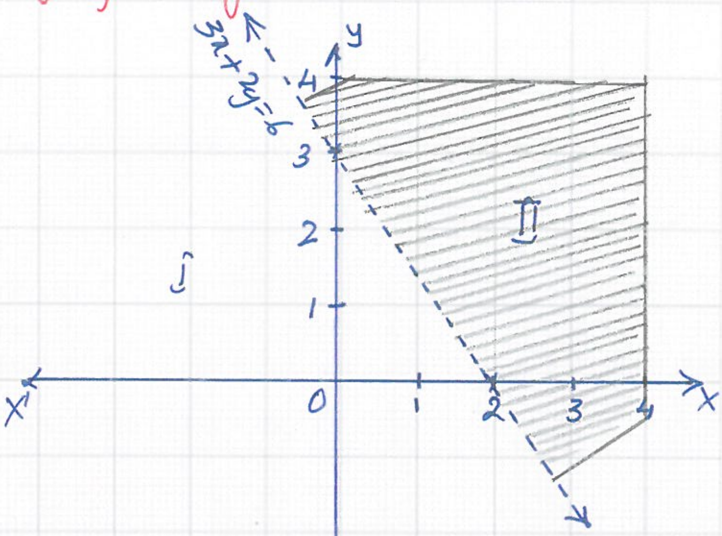
$y=0$; $3x=6$, $x=2$

We select a point (not on the line) (0,0) which lies in one of the half planes.

$3(0)+2(0)>6$

$0>6$, which is false

The shaded half plane II excluding the points on the lines is the solution of the inequality.



Example 2 : Solve $3x-6\geq0$ graphically in 2D plane.

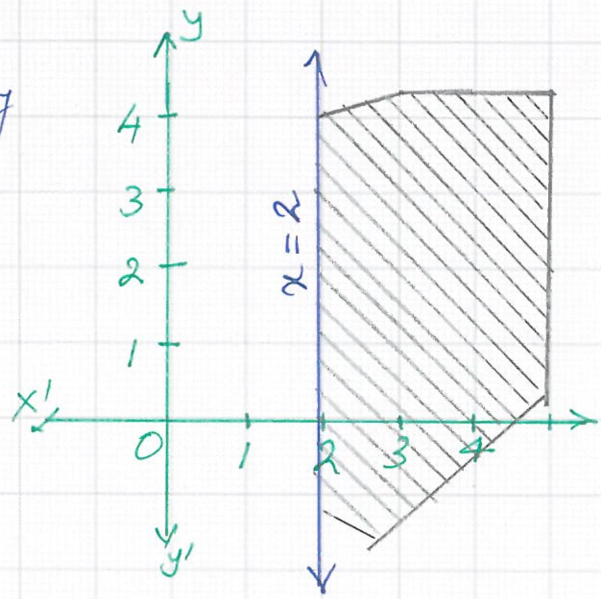
Soln : $3x-6=0$, $x=2$

Select point (0,0) and substituting in inequality $3x-6\geq0$

$0-6\geq0$, $-6\geq0$

which is false

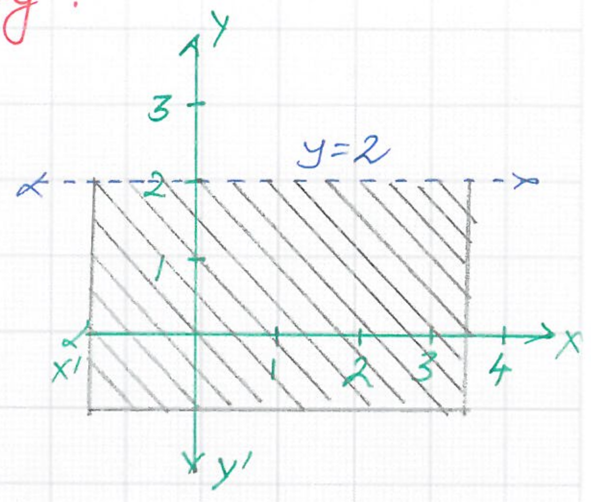
Thus, the solution set is the shaded region on the right hand side of the line $x=2$.



Example 3: Solve $y<2$ graphically.

Sol^n: $y=2$

Select point $(0,0)$ and substitute in inequality $0<2$ which is true. Hence, the solution region is the shaded region below the line $y=2$.



Example 4: Solve the following system of linear inequalities graphically $x+y\geq 5$ and $x-y\leq 3$.

Sol^n: Draw graph of $x+y=5$ and $x-y=3$

$x+y=5$; $x=0$ $y=5$ $(0,5)$

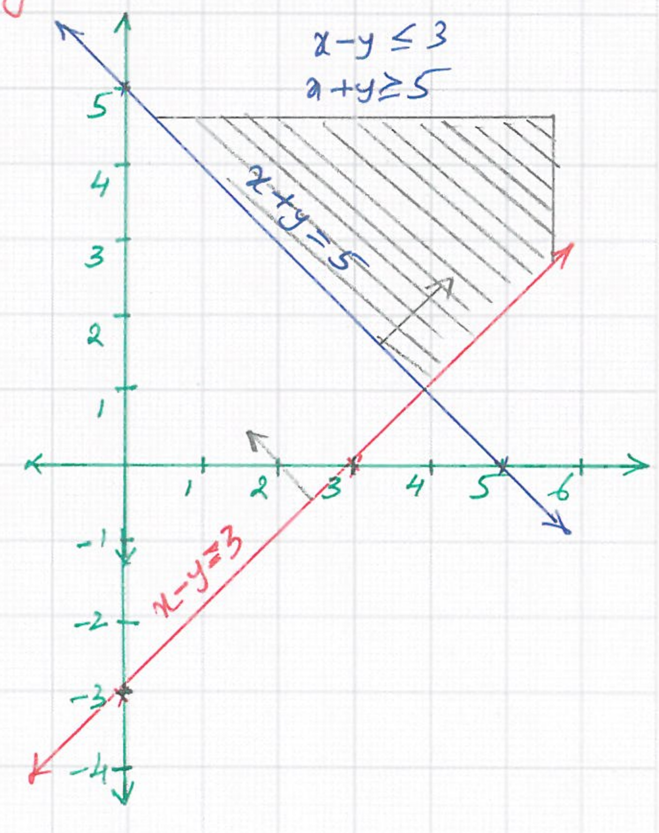
$y=0$ $x=5$ $(5,0)$

$x-y=3$; $x=0$ $y=-3$ $(0,-3)$

$y=0$ $x=3$ $(3,0)$

Select point $(0,0)$ and substitute in both inequalities, we get:

Solution of inequality $x+y\geq 5$ is represented by shaded above $x+y=5$.



Solution of inequality $x-y\leq 3$ is represented by shaded above $x-y=3$.

Region common to both is required region of the given system of inequalities.

Example 5: Solve the following system of inequalities graphically.

$5x+4y \le 40$ ....(1) $\quad x \ge 2$ ...(2) $\quad y \ge 3$ ... (3)

Sol^n: We first draw the graph of lines,

$5x+4y=40$; $\quad x=0,\ y=10 \quad (0,10) \qquad x=2,$ and $y=3$

$\qquad y=0,\ x=8 \ (8,0)$

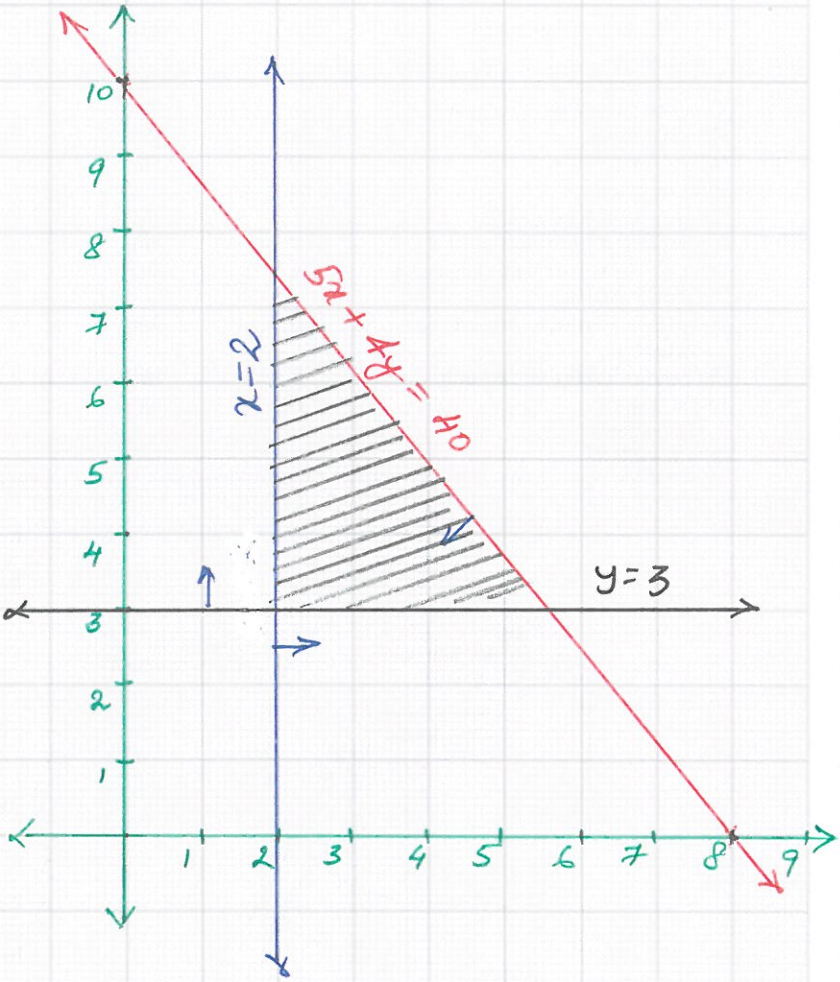
Select point $(0,0)$ and substituting in each inequation.

$0+0 \le 40$ which is true

$0 \ge 2$ which is false

$0 \ge 3$ which is false.

Shaded region including all points on the lines are solutions of the given system of inequalities.



Example 6: Solve the following system of inequalities.

$8x + 3y \le 100 \quad \cdots (1) \qquad x \ge 0 \quad \cdots (2) \qquad y \ge 0 \quad \cdots (3)$

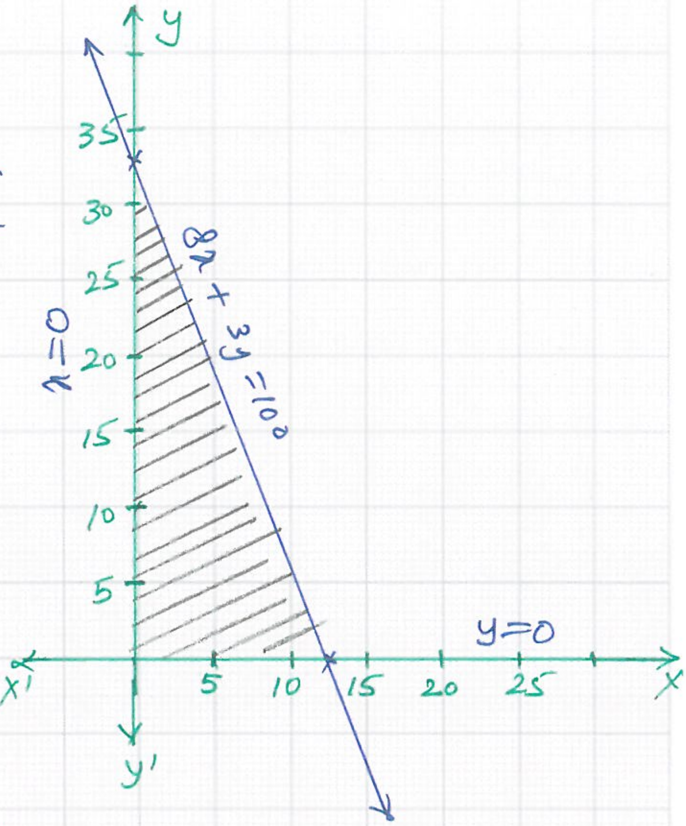
We draw the graph of:

$8x + 3y = 100 \qquad x = 0,\ y = \frac{100}{3}$

$\qquad\qquad y = 0 \quad x = \frac{100}{8} = \frac{25}{2}$

$x = 0,\ y = 0$

$0 + 0 \le 100$ True



Example 15: Solve graphically

$x + 2y \le 8,\ 2x + y \le 8,\ x \ge 0,\ y \ge 0$

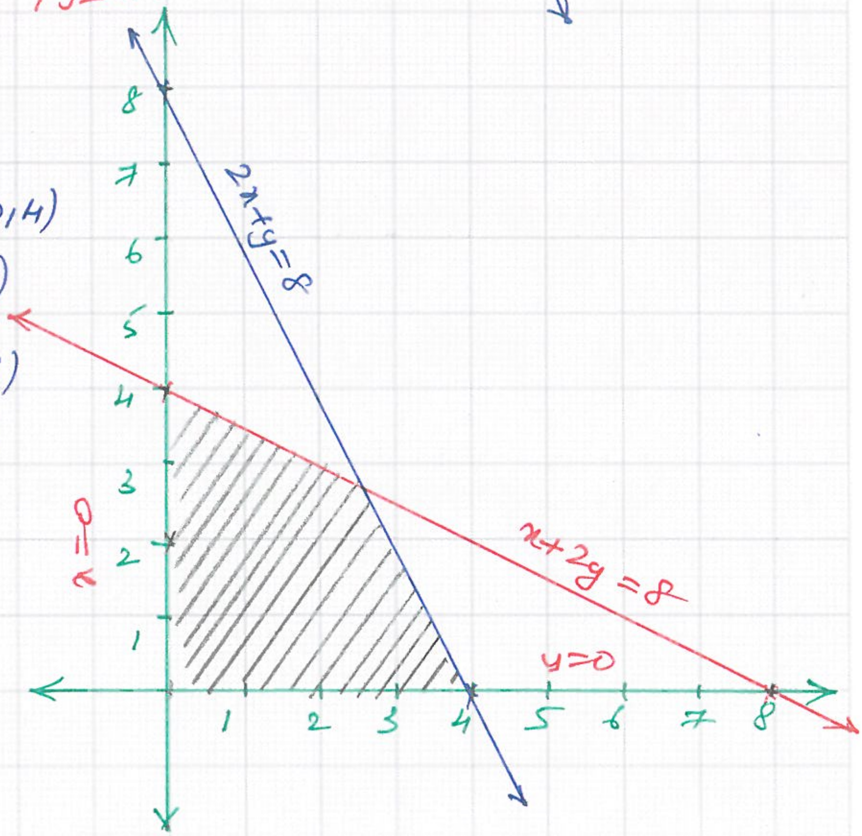
We draw graph of:

$x + 2y = 8;\quad x = 0,\ y = 4 \quad (0,4)$

$\qquad y = 0,\ x = 8 \quad (8,0)$

$2x + y = 8;\quad x = 0 \quad y = 8 \quad (0,8)$

$\qquad y = 0 \quad x = 4 \quad (4,0)$



Exercise

(1) Solve $-8 \leq 5x - 3 < 7$ ANS : $-1 \leq x < 2$

(2) Solve $-5 \leq \frac{5-3x}{2} \leq 8$ ANS : $5 \geq x \geq -\frac{11}{3}$

(3) Solve the following inequalities ANS : $x \geq 2$

$3x - 7 < 5 + x$

$11 - 5x \leq 1$ and represent the solution on number line.

(4) In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by $C = \frac{5}{9}(F-32)$. ANS : $86 < F < 95$.

(5) A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18% ANS : $120 < x < 300$