Chapter 2 - Relations and Functions

Ordered Pair

1) If A be a set and $a, b \in A$, then the ordered pair of elements a and b in A, is denoted by (a, b).
2) An ordered pair (a, b) consists of two elements a and b, where a is the first co-ordinate called abscissa and b the second co-ordinate called ordinate.
3) The ordered pairs (a, b) and (b, a) are different. An ordered pair $(3, 5)$ is not the same as an ordered pair $(5, 3)$.

4) Two ordered pairs (a, b) and (c, d) will be equal if only if $a = c$ and $b = d$.

Cartesian Product of Two Sets

Let A and B be two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$ (read as 'A cross B').

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

1) $A \times B \neq B \times A$ unless $A = B$
2) If $A = B$, then $A \times B = A^2$
3) If one of A and B or both A and B are empty, then $A \times B = \phi$.

4) $n(A \times B) = n(A) \times n(B)$
5) If the set A has m elements and the set B has n elements, then $A \times B$ has mn elements.
6) $A \times B$ and $B \times A$ are equivalent sets.

7) The cartesian product of three sets
$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}.$$

Solved Examples

1) If $\left(\frac{x}{3}+1,\ y-\frac{2}{3}\right) = \left(\frac{5}{3},\ \frac{1}{3}\right)$, find the values of x and y.

Soln: Since the ordered pairs are equal, the corresponding elements are equal.

$\therefore \quad \frac{x}{3}+1 = \frac{5}{3}$ and $y-\frac{2}{3} = \frac{1}{3}$

$\frac{x}{3} = \frac{5}{3}-1 = \frac{2}{3} \Rightarrow x = 2$

$y-\frac{2}{3} = \frac{1}{3},\ y = \frac{1}{3}+\frac{2}{3} = \frac{3}{3} \Rightarrow y = 1$

Hence,
$x = 2$
$y = 1$.

2) If the set A has 3 elements and the set $B = \{3,4,5\}$, then find the number of elements in $(A \times B)$.

Soln: Set $(A \times B)$ has $3 \times 3 = 9$ elements.

3) If $G = \{7,8\}$ and $H = \{5,4,2\}$ find $G \times H$ and $H \times G$.

Soln: $G \times H = \{7,8\} \times \{5,4,2\}$
$= \{(7,5), (7,4), (7,2), (8,5), (8,4), (8,2)\}$

$H \times G = \{5,4,2\} \times \{7,8\}$
$= \{(5,7), (5,8), (4,7), (4,8), (2,7), (2,8)\}$

(4) State true or false, give reason and rewrite correct.

(i) If $P = \{m,n\}$ and $Q = \{n,m\}$, then $P \times Q = \{(m,n), (n,m)\}$

(ii) If A and B are non empty sets, then $A \times B$ is a non empty set of ordered pair (x,y) such that $x \in B$ and $y \in A$.

(iii) If $A = \{1,2\}$ $B = \{3,4\}$ then $A \times (B \cap \phi) = \phi$.

Soln: (i) FALSE

$P = \{m, n\} \quad Q = \{n, m\}$

$P \times Q = \{(m,n), (n,m), (n,n), (m,m)\}$

(ii) FALSE $\quad A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

(iii) $A = \{1, 2\} \quad B = \{3, 4\}$

$B \cap \phi = \phi \quad \therefore A \times (B \cap \phi) = A \times \phi = \phi$.

True.

5) Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$ $C = \{5, 6\}$ $D = \{5, 6, 7, 8\}$

verify that (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times C$ is a subset of $B \times D$.

6) Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Write $A \times B$. How many subsets will $A \times B$ have. List them. [16 sets]

7) If A and B be two non-empty sets, then show that $A \times B = B \times A$ if $A = B$.

Soln: Let $A \times B = B \times A$ and x be any element of A

then $x \in A \Rightarrow (x, b) \in A \times B \quad \forall (b \in B)$

$(x, b) \in B \times A \quad [\because A \times B = B \times A]$ given

$x \in B$

$A \subset B$ ——— (1)

Now let y element of B $\quad y \in B \Rightarrow (a, y) \in A \times B \quad \forall a \in A$

$(a, y) \in B \times A$

$y \in A$

$B \subset A$ ——— (2)

from (1) & (2) $A = B$.

Relations

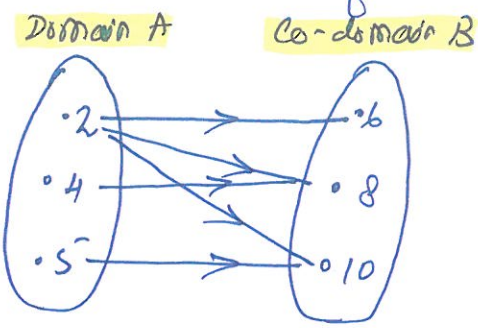
① If A and B are two sets, then a relation from A to B is simply a subset of $(A \times B)$, where
$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

A relation is a set of ordered pair.

So A is called the domain and set B is called the co-domain of the relation.

Let $A = \{2, 4, 5\}$ $B = \{6, 8, 10\}$
$$A \times B = \{(2,6), (2,8), (2,10), (4,6), (4,8), (4,10), (5,6), (5,8), (5,10)\}$$

② Consider the relation "is a factor of". This relation connects the element in set A to the elements in set B since 2 is factor of 8, 5 is factor of 10 and so on.



$R = \{(2,6), (2,8), (2,10), (4,8), (5,10)\}$ represent the relation "is a factor of" from set A to set B.

③ In ordered pair, the second component of an ordered pair is called the image of the corresponding first component. The whole set B is called the co-domain of relation R. Range $\subseteq$ co-domain

If $(a,b) \in R$ we write $a R b$ otherwise $a \not R b$.

④ A relation may be represented algebraically either by the roster method or by the set-builder method.

⑤ An arrow diagram is a visual representation of a relation.

⑥ The number of relations from a set A to a set B is the number of subsets of $A \times B$.

Solved Examples

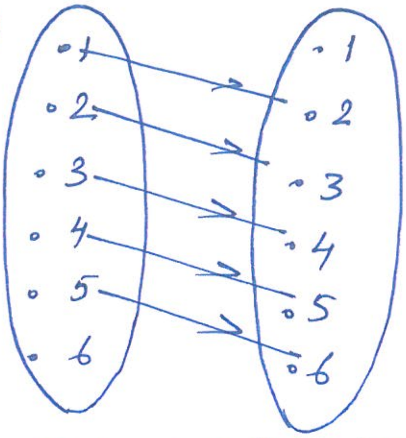
(1) Let $A = \{1, 2, 3, 4, 5, 6\}$ Determine a relation R from A to A by $R = \{(x, y) : y = x+1\}$

(i) Depict this relation using an arrow diagram.
(ii) Write down the domain, co-domain and range R.

Soln: By definition of the relation

$R = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$

(i)

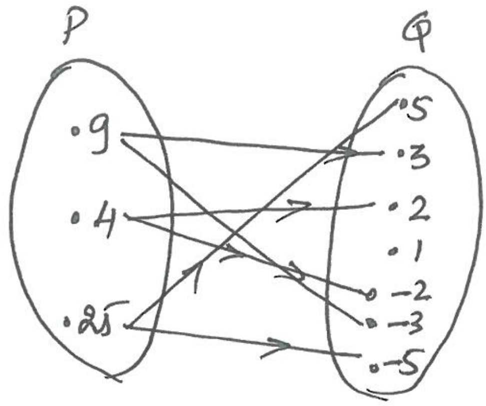


(ii) Domain $= \{1, 2, 3, 4, 5\}$

Range $= \{2, 3, 4, 5, 6\}$

Co-domain $= \{1, 2, 3, 4, 5, 6\}$ (Second set is called co-domain)

(2) In figure, relation between P and Q. Write this relation (i) in set-builder form (ii) in roster form. What is its domain and range?



Soln: It is obvious that the relation R is "x is the square of y".

(i) Set builder form $R = \{(x, y) : x$ is the square of y, $x \in P, y \in Q\}$

(ii) Roster form, $R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$

Domain $= \{9, 4, 25\}$ Range $= \{-2, 2, -3, 3, -5, 5\}$

Note: Element 1 is not related to any element in set P.
The Q is the co-domain of this relation.

(3) $A = \{1,2\}$ $B = \{3,4\}$. Find the number of relations A to B.

Soln : $A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$n(A \times B) = 4$

No. of subsets of $A \times B = 2^4$

$\therefore$ The no. of relations from A to B will be 2^4.

Universal Relation : A relation R on a set A is called the universal relation if $R = A \times A$.

If $A = \{1, 2, 3\}$

$R = A \times A = \{(1,1), (1,2), (1,3) \ldots\}$ is universal relation in A.

Null Relation : Since every subset of $A \times A$ is a relation on A and is also a subset of $A \times A$ and as such the null set ϕ is a relation on A which is called the void relation on A.

As $\phi \subseteq A \times B$, so ϕ is a relation from A to B

Domain $\phi = \phi$ and range $\phi = \phi$.

Identity Relation on a Set : The identity relation on a set A is the set of all ordered pairs $(a, b) \in A \times A$ such that $a = b$. Denoted by I_A

$I_A = \{(a, b) : a \in A, b \in A \text{ and } a = b\}$

If $A = \{1, 2, 3, 4, 5, 6\}$ $I_A = \{(1,1), (2,2), (3,3) \ldots\}$

Inverse Relation : The inverse relation (R^{-1}) is relation from B to A.

$R^{-1} = \{(b, a) : (a, b) \in R\}$. R^{-1} consists of ordered pair when reversed, belong to R.

Functions

A function is a special relation : a set of ordered pairs in which no two distinct ordered pairs have the same first element.

In the relation $y = x^2$ $(2,4), (-2,4), (3,9), (-3,9) \ldots$
For each value of x there is only one corresponding value of y.
Therefore, $y = x^2$ is a function.

In the relation $x = y^2$ $(4,2), (4,-2), (9,3), (9,-3) \ldots$
We see that two ordered pairs have same first element.
Therefore, $x = y^2$ is not a function.

for example, $A = \pi r^2$, the area of circle depends on its radius.
We say that "A is a function of r".

Definition:

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.
If f is a function from A to B and $(a,b) \in f$, then $f(a) = b$, where "b" is called the image of "a" under f and a is called the pre-image of "b" under f.

Let $f: A \rightarrow B$, then set A is called domain
set B is called co-domain.

Example's

$A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 5, 6, 11, 12, 18\}$
A function $f: A \rightarrow B$ is defined by $f(x) = x^2+2$,
$f(1) = 3$, $f(2) = 6$ $f(3) = 11$ etc.

(i) $f = \{(1,3), (2,6), (3,11), (4,18)\}$

(ii) Domain of $f = A$, Co-Domain of $f = B$

(iii) Range of $f = \{3, 6, 11, 18\}$

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Solved Examples

(1) Which of the following relations are function? Give reason. If it is a function, determine its domain and range.

(i) $\{(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)\}$

(ii) $\{(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)\}$

(iii) $\{(1,3)\ (1,5)\ (2,5)\}$

(iv) $\{(0,0), (1,1), (1,-1), (4,2), (9,3), (9,-3), (16,4), (16,-4)\}$

Soln: (i) It is a function

Range of $f = \{1\}$

Domain of $f = \{2, 5, 8, 11, 14, 17\}$

(ii) It is a function

Domain of $f = \{2, 4, 6, 8, 10, 12, 14\}$

Range of $f = \{1, 2, 3, 4, 5, 6, 7\}$

(iii) Not a function

(iv) Not a function

(2) Let $A = \{9, 10, 11, 12, 13\}$ and let $f: A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

Soln $f(9) = 3$ $f(10) = 5$ $f(11) = 11$ $f(12) = 3$

Range of f = The set of values of $f(n) = \{3, 5, 11, 13\}$

(3) Find the range of each of the following functions given by.

(i) $f(x) = 2-3x$, $x \in R$, $x > 0$

(ii) $f(x) = x^2+2$, x is a real number

(iii) $f(x) = x$, x is a real number.

Sol^n: (i) $y = f(x) = 2-3x \Rightarrow 3x = 2-y$

$x = \dfrac{2-y}{3}$ $\quad x \in R,\ x > 0 \quad \dfrac{2-y}{3} > 0$

$\Rightarrow \quad 2-y > 0 \qquad y < 2$

Range of $f = \{y : y < 2,\ y \in R\} = (-\infty, 2)$.

(ii) Let $y = f(x) = x^2+2$; $x^2 = y-2$; $x = \sqrt{y-2}$

For x to be real $y-2 > 0 \quad y > 2$

Range of $f = [2, \infty)$

(iii) Let $y = f(x) = x \qquad y = x$

Range of $f = R$ (all real values).

(4) Find the domain of the function $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$.

Sol^n: $x^2+2x+1 = x^2+x+x+1 = x(x+1)+1(x+1) = (x+1)(x+1)$

$x^2-8x+12 = x^2-6x-2x+12 = x(x-6)-2(x-6) = (x-2)(x-6)$

$x^2-8x+12 = (x-2)(x-6)$

$f(x)$ is defined for all real numbers except $x = 2$, $x = 6$

Hence, Domain of $f = R - \{2, 6\}$

(5) Find the domain and range of the real function $f(x) = -|x|$.

Soln : $f(x) = -|x|$ x can have all real values.

$f(x)$ is defined for all $x \in R$.

Domain of $f = R$.

Again $|x|$ can never be negative

$-|x| \leq 0$

Range of $f = (-\infty, 0]$.

(6) Find the domain and the range of the real function f defined by $f(x) = \sqrt{x-1}$

Soln : f is only defined for real values of x which $x > 0$

Thus domain of f is the set of all real numbers for which $x - 1 > 0$, $x > 1$ $[1, \infty)$

Let $y = f(x) = \sqrt{x-1}$ $y = \sqrt{(x-1)}$ for $x \geq 1$ $y \geq 0$

Range of $f = [0, \infty)$

(7) Find the domain and range of the real function f defined by $f(x) = |x-1|$.

Soln : $f(x) = |x-1|$

The domain of f is the set of all real numbers since $|x-1| > 0$

Domain $(f) = R$

Let $y = f(x) = |x-1|$; $y = |x-1|$

$\therefore$ The range of f is the set of non-negative real numbers.

Range $(f) = [0, \infty)$.

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8) Find the domain and range of the following functions:

(i) $f = \{(x, \sqrt{9-x^2}) : x \in R\}$

(ii) $f = \{x, \frac{x^2-1}{x-1} : x \in R, x \neq 1\}$

Soln: (i) We define $f(x) = \sqrt{9-x^2}$

$\sqrt{9-x^2}$ is defined for all those values of $x \in R$ for which $9-x^2 \geq 0$ or $x^2 < 9$ or $-3 < x < 3$

$(3-x)(3+x) \geq 0$

$\therefore$ Domain of $f : [x : x \in R$ and $-3 < x < 3\}$

$= [-3, 3]$

To find range, let $y = f(x) = \sqrt{9-x^2}$; $y^2 = 9-x^2$

$x^2 = 9-y^2$ $\quad x^2 \geq 0 \quad x \in R.$ $\quad 3-y \geq 0 \quad y \geq 3$

$\therefore 9-y^2 \geq 0$ $\quad 3+y \geq 0 \quad y \geq -3$

$(3-y)(3+y) \geq 0$

$\Rightarrow -3 < y < 3$

But $y \geq 0$ $\therefore$ $y \in [0, 3]$.

Hence, range of f is closed interval $[0, 3]$.

(ii) $y = \frac{x^2-1}{x-1} : x \in R \; x \neq 1$ $\quad y = x+1$ if $x \neq 1$

Since $\frac{x^2-1}{x-1}$ is not defined only when $x = 1$

Domain of $\frac{x^2-1}{x-1}$ is $R - \{1\}$

to get $y = 2$ x must be 1 which is absurd.

$\therefore y$ takes all real values except 2

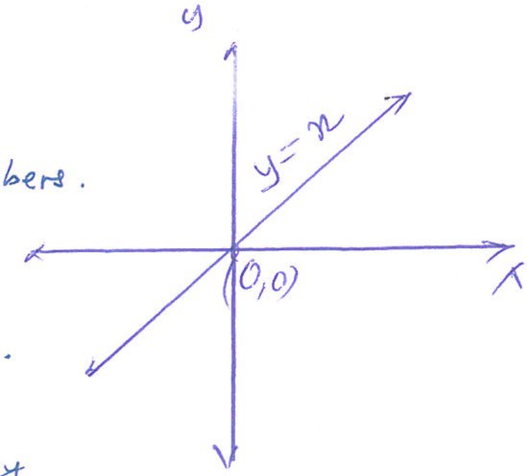
Range of $\frac{x^2-1}{x-1}$ is $R - \{2\}$.

Graph of functions

The graph of a function f is the set of points $[x, f(x)]$ such that x is in the domain of f. The y-coordinate of any point (x, y) on the graph is $y = f(x)$.

On the graph of a function, we cannot have two points (x_1, y_1) and (x, y_2) with the same abscissa x and different ordinates y_1 and y_2.

1) Identity function :



Let R be the set of the real numbers. The real valued function $f: R \to R$ defined by $y = f(x) = x$ for each $x \in R$ is called the identity function.

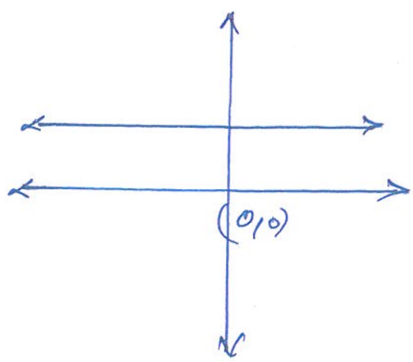
The domain and range of identity function f are both equal.

The graph of the identity function f is a straight line and passes through the origin.

Identity function $f(x) = x$ Domain $(f) = (-\infty, \infty)$ Range$(f) = (-\infty, \infty)$

2) Constant function

$f: R \to R$ by $y = f(x) = c$ $x \in R$ where c is a constant and each $x \in R$ is called a constant function.



The domain of f is R and its range is $\{c\}$.

Example, $f(x) = 3$ constant function

$f(-1) = 3$, $f(0) = 3$, $f(5) = 3$ so on.

The graph of a constant function is a line parallel to x-axis.

(3) Polynomial function

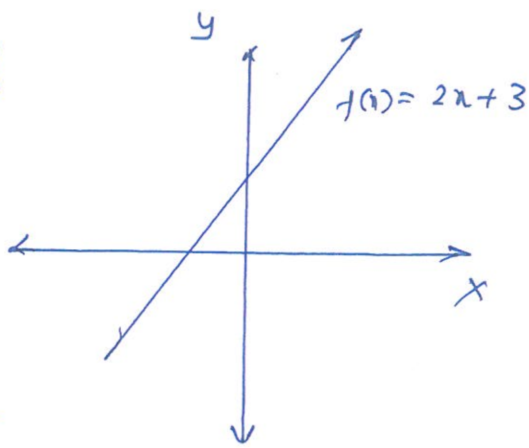
A function $f : R \rightarrow R$ is said to be polynomial function if for each x in R,
$y = f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$.

where n is a non-negative integer.
$a_0, a_1, a_2, \dots a_n \in R$ are all constant.

Examples, $f(x) = 2x+3$, $g(x) = x^2$ $h(x) = x^3$ etc.

(i) Linear function:

A linear function is a polynomial of degree 1. A function that can be expressed in the form $f(x) = mx+b$, where m and b are real numbers with $m \neq 0$ is called a linear function.
The graph of a linear function is a straight line.



(ii) Quadratic Function: take the general form $y = ax^2 + bx + c$

(iii) Cubic Function: functions where the highest power is cubed (x^3) are called cubic functions. There shape when plotted is known as a cubic curve.

(4) Rational Functions: The function of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials functions of x, defined in a domain, provided $g(x) \neq 0$ are called rational function.
The domain of a rational function is restricted to real numbers that do not cause the denominator to have a value of 0.

(5) The Modulus Function

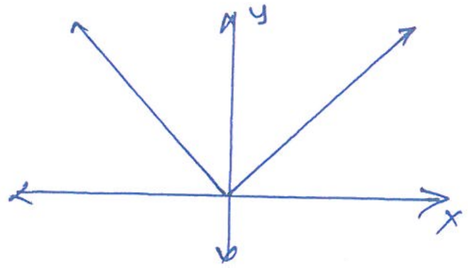
The symbol $|x|$ is read as "the absolute value of x".

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The function $f : R \to R$ defined by $f(x) = |x|$ for each $x \in R$ is called modulus function.

This function can be written as $f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

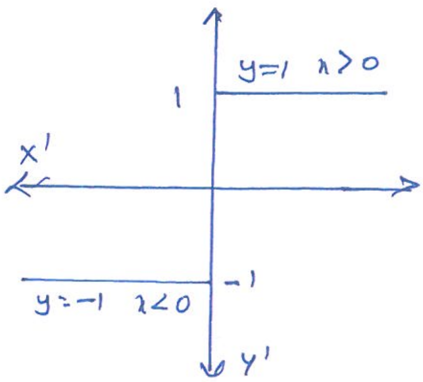
Graph of function $f(x) = |x|$



(6) The Signum Function

The function $f : R \to R$ defined by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ is called the signum function.

The domain of the signum function is R and the range is the set $\{-1, 0, 1\}$.



(7) The greatest integer function :

The function $f : R \to R$ defined by $f(x) = [x]$, $x \in R$ is called the greatest integer function.

The symbol $[x]$ is defined to be the greatest or largest integer less than or equal to x. for example $[3.01] = 3$, $[5.2] = 5$ $[-4.2] = -5$ $[6] = 6$

Algebra of Real function.

Let f and g be two functions.

(i) Sum : $(f+g)(x) = f(x)+g(x)$

(ii) Difference : $(f-g)(x) = f(x)-g(x)$

(iii) Product : $(fg)(x) = f(x)\cdot g(x)$

(iv) Quotient : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Solved Examples:

1) Let $f, g : R \to R$ be defined by $f(x) = x+1$, $g(x) = 2x-3$. Find $f+g$, $f-g$ and $\frac{f}{g}$. $\left[3x-2,\ -x+4,\ \frac{x+1}{2x-3},\ x \neq 3/2\right]$

2) Find the domain of $f(x) = \sqrt{x-1} + \sqrt{6-x}$.

Sol^n: Let $f_1(x) = \sqrt{x-1}$ which exist for $x \geq 1$.
Again $f_2(x) = \sqrt{6-x}$ which exist for $x \leq 6$.

Domain of $f = [1,\infty) \cup (-\infty, 6] = [1, 6]$.

3) Find the range of the function $f(x) = \frac{x+2}{|x+2|}$ $x \neq -2$.

Sol^n: $y = f(x) = \frac{x+2}{|x+2|}$ $y = \frac{x+2}{|x+2|}$ $x \neq -2$.

If $x+2 > 0$ $y = \frac{x+2}{x+2} = 1$

If $x+2 < 0$ $y = \frac{x+2}{-(x+2)} = -1$

$\therefore$ Range of $f = \{-1, 1\}$.

4) If $f: R \rightarrow R$ is defined by $f(x) = 3x + |x|$
Prove that $f(2x) - f(-x) - 6x = f(x)$

Solⁿ:
$$f(x) = 3x + |x|$$
$$f(2x) = 3(2x) + |2x| = 6x + 2|x|$$
$$f(-x) = 3(-x) + |-x| = -3x + |x| \qquad [|-x| = |x|]$$

$$f(2x) - f(-x) - 6x = 6x + 2|x| - (-3x + |x|) - 6x$$
$$= \cancel{6x} + 2|x| + 3x - |x| - \cancel{6x}$$
$$= 3x + |x|$$
$$= f(x).$$

5) If $f(x) = \frac{x-1}{x+1}$ then show that $f(\frac{1}{x}) = -f(x)$

Solⁿ: $f(x) = \frac{x-1}{x+1}$ $\quad f(\frac{1}{x}) = \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = \frac{1-x}{1+x}$

$$f(x) = \frac{-(x-1)}{x+1} = -f(x)$$

6) If $f(x) = y = \frac{ax-b}{(cx-a)}$, then prove that $f(y) = x$

Solⁿ: $y = \frac{ax-b}{cx-a}$,
$$y(cx-a) = ax-b$$
$$ycx - ya = ax-b$$
$$ycx - ax = ya-b$$
$$x(yc-a) = ya-b$$
$$x = \frac{ay-b}{cy-a}$$
$$x = f(y). \quad \text{or} \quad f(y) = x.$$