CONIC SECTION

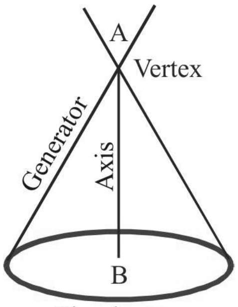
1. CONIC SECTIONS

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.
- The line passing through the focus & perpendicular to the directrix is called the **Axis.**
- A point of intersection of a conic with its axis is called a **Vertex.**

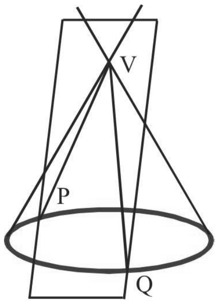
2. SECTION OF RIGHT CIRCULAR CONE BY DIFFERENT PLANES

A right circular cone is as shown in the fig. 1



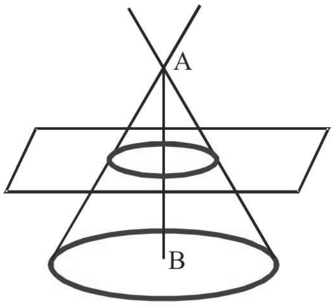
Circular Base
(Fig. 1)

(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in fig. 2



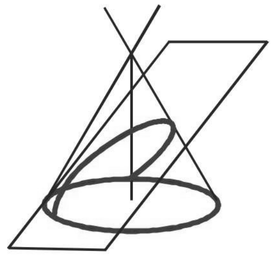
(Fig. 2)

(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the fig. 3.



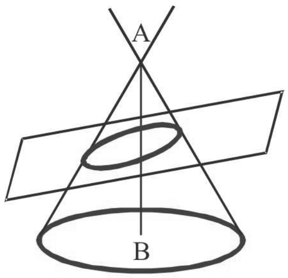
(Fig. 3)

(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the fig. 4.

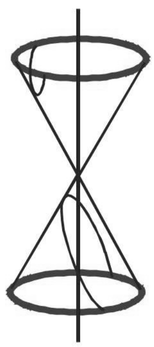


(Fig. 4)

(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the fig. 5 & 6.

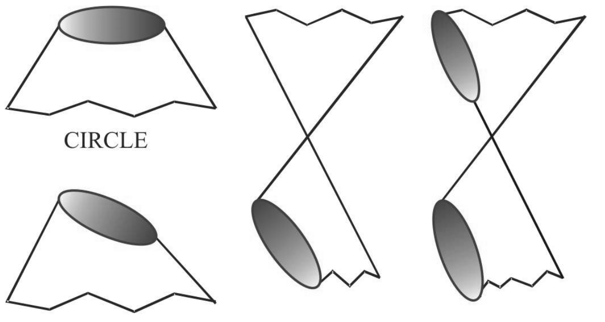


(Fig. 5)



(Fig. 6)

3 D View :



3. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$(l^2 + m^2)[(x - p)^2 + (y - q)^2] = e^2(lx + my + n)^2 \equiv$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

4. DISTINGUISHING VARIOUS CONICS

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) When The Focus Lies On The Directrix.

In the case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if :

$e > 1 \equiv h^2 > ab$, the lines will be real and distinct intersecting at S.

$e = 1 \equiv h^2 \geq ab$, the lines will coincident.

$e < 1 \equiv h^2 < ab$, the lines will be imaginary.

Case (II) When The Focus Does Not Lie On Directrix.

a parabola : $e = 1$, $\Delta \neq 0$, $\mathbf{h^2 = ab}$

an ellipse : $0 < e < 1$; $\Delta \neq 0$, $\mathbf{h^2 < ab}$

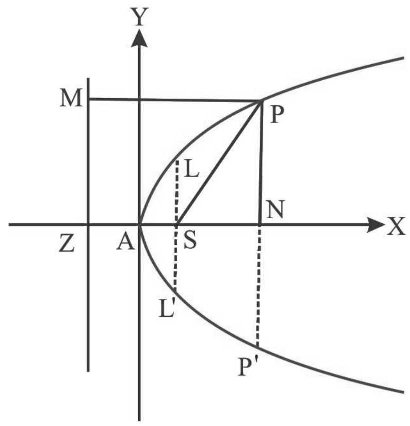
a hyperbola : $e > 1$; $\Delta \neq 0$, $\mathbf{h^2 > ab}$

rectangular hyperbola : $e > 1$; $\Delta \neq 0$, $\mathbf{h^2 > ab; a + b = 0}$

PARABOLA

5. DEFINITION AND TERMINOLOGY

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix).



Four standard forms of the parabola are

$\mathbf{y^2 = 4ax \ ; \ y^2 = -4ax \ ; \ x^2 = 4ay \ ; \ x^2 = -4ay}$

For parabola $y^2 = 4ax$:

(i) Vertex is (0, 0) (ii) Focus is (a, 0)

(iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

Focal Distance : The distance of a point on the parabola from the focus.

Focal Chord : A chord of the parabola, which passes through the focus.

Double Ordinate : A chord of the parabola perpendicular to the axis of the symmetry.

Latus Rectum : A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a

$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, –2a).

Note...

(i) Perpendicular distance from focus on directrix = half the latus rectum

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

6. POSITION OF A POINT RELATIVE TO A PARABOLA

The point $(x_1\ y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $\mathbf{y_1^2 - 4ax_1}$ is **positive**, **zero** or **negative**.

7. LINE & A PARABOLA

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in :

- two real points if $a > mc$
- two coincident points if $a = mc$
- two imaginary points if $a < mc$

$\Rightarrow$ **condition of tangency is, $c = a/m$.**

Length of the chord intercepted by the parabola on the line $y = mx + c$ is :

$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$$

Note...

Length of the focal chord making an angle α with the x-axis is $4a\,\mathrm{cosec}^2\alpha$.

8. PARAMETRIC REPRESENTATION

The simplest & the best form of representing the co-ordinates of a point on the parabola is $\mathbf{(at^2, 2at)}$ i.e. the equations $\mathbf{x = at^2}$ & $\mathbf{y = 2at}$

together represents the parabola $y^2 = 4ax$, t being the parameter.

The equation of a chord joining t_1 & t_2 is

$\mathbf{2x - (t_1 + t_2)y + 2at_1t_2 = 0}$.

Note...

If t_1 and t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $\mathbf{t_1t_2 = -1}$.

Hence the co-ordinates at the extremities of a focal chord can be taken as : $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

9. TANGENTS TO THE PARABOLA $y^2 = 4ax$

(i) $\mathbf{y y_1 = 2a(x + x_1)}$ at the point (x_1, y_1) ;

(ii) $\mathbf{y = mx + \frac{a}{m}}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $\mathbf{t y = x + at^2}$ at point $(at^2, 2at)$.

Note...

Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$.

10. NORMALS TO THE PARABOLA $y^2 = 4ax$

(i) $\mathbf{y - y_1 = \frac{-y_1}{2a}}(x - x_1)$ at (x_1, y_1)

(ii) $\mathbf{y = mx - 2am - am^3}$ at point $(am^2, -2am)$

(iii) $\mathbf{y + tx = 2at + at^3}$ at point $(at^2, 2at)$.

Note...

(i) Point of intersection of normals at t_1 & t_2 are, $a(t_1^2 + t_2^2 + t_1t_2 + 2)$; $-at_1 t_2 (t_1 + t_2)$

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1 meets the parabola again at the point t_2 then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

11. PAIR OF TANGENTS

The equation to the pair of tangents which can be drawn from any point $(x_1\ y_1)$ to the parabola $y^2 = 4ax$ is given by: $\mathbf{SS_1 = T^2}$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y y_1 - 2a(x + x_1)$.

12. DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to a curve is called the director circle. For parabola $y^2 = 4ax$ it's equation is

$x + a = 0$ which is parabola's own directrix.

13. CHORD OF CONTACT

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $\mathbf{yy_1 = 2a (x + x_1)}$; (i.e., T = 0)

Note...

The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $(y_1^2 - 4ax_1)^{3/2} \div 2a$.

14. CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is : $(x_1\ y_1)$ is :

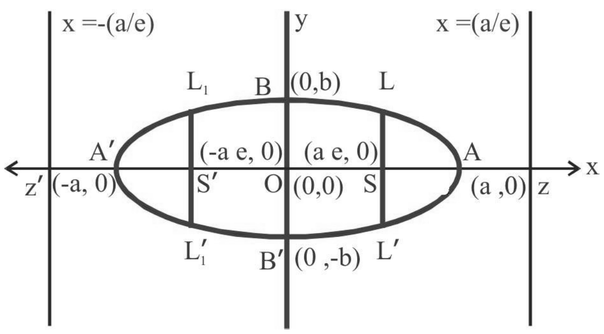
$$\mathbf{y - y_1 = \frac{2a}{y_1}(x - x_1) \equiv T = S_1.}$$

ELLIPSE

15. STANDARD EQUATION AND DEFINITIONS

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \quad a > b \quad \& \quad b^2 = a^2(1 - e^2).$$



Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

Foci : $S \equiv (a\,e, 0)$ & $S' \equiv (-a\,e, 0)$

Equations of Directrices : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$

Major Axis : The line segment A'A in which the foci S' & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0, -b)$ and $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major and minor axis together are called principal axis of the ellipse.

Vertices : $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

Length of latus rectum (LL') :

$$\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1 - e^2)$$

= 2e (distance from focus to the corresponding directrix).

Centre :

The point which bisects every chord of the conic drawn through it is called the centre of the conic. $C \equiv (0, 0)$

the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

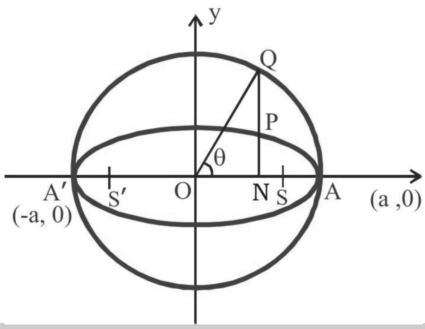
Note...

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned then the rule is to assume that a > b.

(ii) If b > a is given, then the y-axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.

16. AUXILIARY CIRCLE/ECCENTRIC ANGLE

A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the **Corresponding Points** on the ellipse and the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse ($-\pi < \theta \le \pi$).



Note...

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi major axis}}{\text{Semi major axis}}$$

Note...

If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

17. PARAMETRIC REPRESENTATION

The equation $\mathbf{x = a\cos\theta}$ **&** $\mathbf{y = b\sin\theta}$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Where θ is a parameter. Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then;

$Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α and β is given by

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}.$$

18. POSITION OF A POINT W.R.T. AN ELLIPSE

The point p (x_1, y_1) lies outside, inside or on the ellipse according as ;

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0 \quad \text{(outside)}$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0 \quad \text{(inside)}$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0 \quad \text{(on)}$$

19. LINE AND AN ELLIPSE

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is $<\ =$ or $>\ a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $\mathbf{c^2 = a^2m^2 + b^2}$.

20. TANGENTS

(a) Slope form $\mathbf{y = mx \pm \sqrt{a^2m^2 + b^2}}$ is tangent to the ellipse for all values of m.

(b) Point form $\mathbf{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$ is tangent to the ellipse at $(x_1\ y_1)$.

(c) Parametric for $\mathbf{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

Note...

(i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction. These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is :

$$\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

(iii) The eccentric angles of point of contact of two parallel tangents differ by π.

21. NORMALS

(i) Equation of the normal at $(x_1\ y_1)$ is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2.$$

(ii) Equation of the normal at the point ($a\cos\theta$, $b\sin\theta$) is;

$$ax\sec\theta - by\operatorname{cosec}\theta = (a^2 - b^2).$$

(iii) Equation of a normal in terms of its slope 'm' is

$$y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$

22. DIRECTOR CIRCLE

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

Note...

Pair of tangents, Chord of contact, Pole & Polar, Chord with a given Middle point are to be interpreted as they are in Parabola/Circle.

23. DIAMETER (NOT IN SYLLABUS)

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation $y = -\frac{b^2}{a^2m}x$

Note...

All diameters of ellipse passes through its centre.

IMPORTANT HIGHLIGHTS

Referring to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{2} = 1$

(a) If P be any point on the ellipse with S & S' as to foci then l(SP) + l(S'P) = 2a.

(b) The tangent & normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

(c) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.

(d) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

(e) If the normal at any point P on the ellipse with centre C meet the major and minor axes in G & g respectively & if CF be perpendicular upon this normal then :

(i) PF. PG = b^2

(ii) PF. Pg = a^2

(iii) PG. Pg = SP. S'P

(iv) CG. CT = CS^2

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[Where S and S' are the foci of the ellipse and T is the point where tangent at P meet the major axis]

* The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

* If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then :

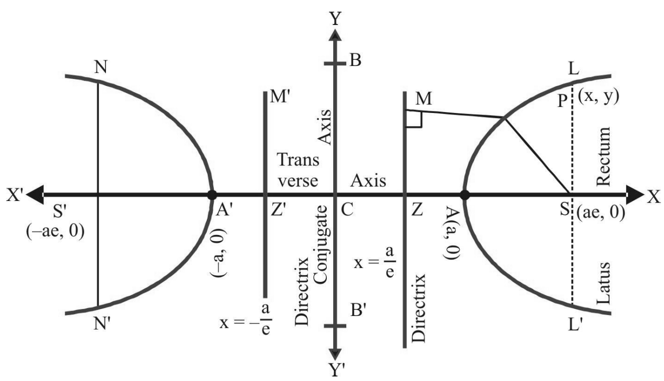
(i) T t. PY = $a^2 - b^2$ and

(ii) least value of T t is a + b.

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater thatn unity (e > 1).

24. STANDARD EQUATION & DEFINITION (S)



Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

where $b^2 = a^2 (e^2 - 1)$.

Eccentricity (e) : $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{C.A}{T.A}\right)^2$

(C.A $\to$ Conjugate Axis;
T.A $\to$ Transverse Axis)

Foci : S $\equiv$ (ae, 0) & S' $\equiv$ (–ae, 0).

Equations of Directrix : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$

Transverse Axis : The line segment A'A of length 2a in which the foci S' & S both lie is called the transverse axis of the hyperbola.

Conjugate Axis : The line segment B'B between the two points B' $\equiv$ (0, –b) & B $\equiv$ (0, b) is called as the conjugate axis of the hyperbola.

Principal Axes : The transverse & conjugate axis together are called Principal Axes of the hyperbola.

Vertices : A $\equiv$ (a, 0) & A' $\equiv$ (–a, 0)

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the transverse axis is called a double ordinate.

Latus Rectum (l) L : The focal chord perpendicular to the transverse axis is called the latus rectum.

$$\ell = \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a\,(e^2 - 1)$$

Note...

l (L.R.) = 2 e (distance from focus to directrix).

Centre : The point which bisects every chord of the conic drawn through it is called the centre of the conic. C $\equiv$ (0, 0) the origin is the centere of the hyperbola

$$\frac{x}{a} - \frac{y}{b} = 1$$

Note...

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

25. RECTANGULAR OR EQUILATERAL HYPERBOLA

The particular kind of hyperbola in which the lengths of the transverse and conjugate axis are equal is called an Equilateral Hyperbola. Note that the **eccentricity of the rectangular hyperbola is $\sqrt{2}$.**

26. CONJUGATE HYPERBOLA

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called Conjuagate Hyperbolas of each other.

e.g. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are **conjugate hyperbolas** of each other.

Note...

(a) If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

(d) Two similiar hyperbolas are said to be equal if they have same latus rectum.

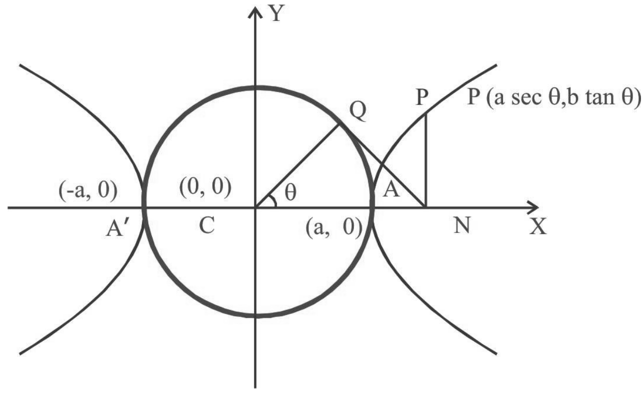
(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

27. AUXILIARY CIRCLE

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola and the auxiliary circle.

In the hyperbola any ordinate of the curve does not meet the circle on AA' as diameter in real points. There is therefore no real eccentric angle as in the case of the ellipse.



28. PARAMETRIC REPRESENTATION

The equation **x = a sec θ & y = b tan θ** together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

Note that if $P(\theta) \equiv (a \sec\theta, b \tan \theta)$ is on the hyperbola then ;

$Q(\theta) \equiv (a \cos \theta, a \tan \theta)$ is on the auxiliary circle.

The equation to the chord of the hyperbola joining two points with eccentric angles α and β is given by

$$\frac{x}{a}\cos\frac{-\beta}{2} - \frac{y}{b}\sin\frac{+\beta}{2} = \cos\frac{\alpha+\beta}{2}.$$

29. POSITION OF A POINT 'P' W.R.T. A HYPERBOLA

The quantity $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies inside, on or outside the curve.

30. LINE AND A HYPERBOLA

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 >$ or $=$ or $< a^2m^2 - b^2$, respectively.

31. TANGENTS

(i) **Slope Form :** $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) **Point Form :** Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_1\ y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(iii) **Parametric Form :** Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec \theta, b \tan \theta)$ $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note...

(i) Point of intersection of the tangents at θ_1 & θ_2 is :

$$x = a\frac{\cos\frac{\theta_1 - \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}}, y = b\tan\left(\frac{\theta_1 + \theta_2}{2}\right)$$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touches the hyperbola at the extremities of a diameter.

32. NORMALS

(i) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P (x_1, y_1) on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$

(ii) The equation of the normal at the point P (a sec θ, b tan θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$.

Note...

Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in parabola/circle.

33. DIRECTOR CIRCLE

The locus of the intersection point of tangents which are at right angles is known as the Director Circle of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real.

If $b^2 = a^2$ (rectangular hyperbola) the radius of the circle is zero and it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no pair of tangents at right angle can be drawn to the curve.

34. DIAMETER (NOT IN SYLLABUS)

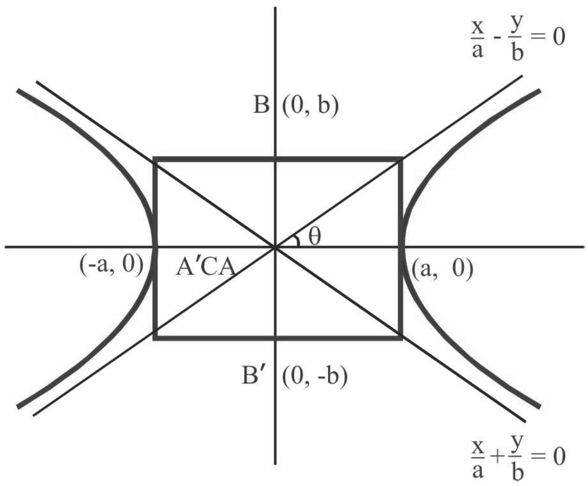
The locus of the middle points of a system of parallel chords with slope 'm' of an hyperbola is called its diameter. It is a straight line passing through the centre of the hyperbola and has the equation $y = +\frac{b^2}{a^2m}x$

Note...

All diameters of the hyperbola passes through its centre.

35. ASSYMPTOTES (NOT IN SYLLABUS)

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the hyperbola.



Equation of Asymptote : $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0$

Pairs of Asymptotes : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

Note...

(i) A hyperbola and its conjugate have the same asymptote.

(ii) The equation of the pair of asymptotes different form the equation of hyperbola (or conjugate hyperbola) by the constant term only.

(iii) The asymptotes pass through the centre of the hyperbla and are equally inclined to the transverse axis of the hyperbola. Hence the bisectors of the angles between the asymptotes are the principle axes of the hyperbola.

(iv) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(v) Asymptotes are the tangent to the hyperbola from the centre.

(vi) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as :

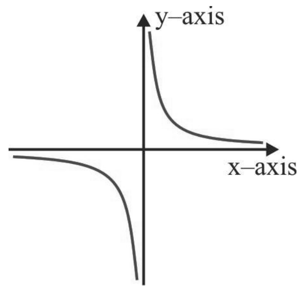
Let f(x y) = 0 represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

36. RECTANGULAR HYPERBOLA ($xy = c^2$)

It is referred to its asymptotes as axes of co-ordinates.

Vertices : (c, c) and (–c, –c) ;



Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$,

Directrices : $x + y = \pm \sqrt{2}c$

Latus Rectum (l) : $l = 2\sqrt{2}c$ = T.A. = C.A.

Parametric equation $\mathbf{x = ct, y = c/t}$, $t \in R - \{0\}$

Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $\mathbf{x + t_1t_2y = c(t_1 + t_2)}$.

Equation of the tangent at P $(x_1\ y_1)$ is $\mathbf{\frac{x}{x_1}+\frac{y}{y_1}=2}$ and at P(t) is $\mathbf{\frac{x}{t} + ty = 2c}$.

Equation of the normal at P(t) is $xt^3 - yt = c(t^4 - 1)$.

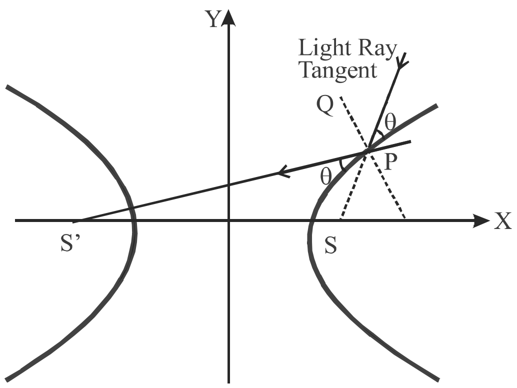
Chord with a given middle point as (h, k) is kx + hy = 2hk.

IMPORTANT HIGHLIGHTS

(i) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is b^2.

(ii) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(iii) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spell the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}+\frac{y^2}{k^2-b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(iv) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

(v) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi conjugate axis.

(vi) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.

(vii) The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself and for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

(viii) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.

(ix) A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle.

If $\left(ct_i, \frac{c}{t_i}\right)$ $i = 1, 2, 3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1\,t_2\,t_3\right)$.

(x) If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points, t_1, t_2, t_3 and t_4, then

(a) $t_1\,t_2\,t_3\,t_4 = 1$

(b) the centre of the mean position of the four points bisects the distance between the centre of the circle through the points t_1, t_2, and t_3 is :

$$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1+t_2+t_3\right)\right\}$$

CIRCLE

A circle is a locus of a point whose distance from a fixed point (called centre) is always constant (called radius).

1. EQUATION OF A CIRCLE IN VARIOUS FORM

(a) The circle with centre as origin & radius 'r' has the equation; $\mathbf{x^2 + y^2 = r^2}$.

(b) The Circle with centre (h, k) & radius 'r' has the equation; $\mathbf{(x - h)^2 + (y - k)^2 = r^2}$

(c) The general equation of a circle is

$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$ with centre as $(-g, -f)$ and

radius = $\sqrt{g^2 + f^2 - c}$. If :

$g^2 + f^2 - c > 0 \quad \Rightarrow \quad$ real circle.

$g^2 + f^2 - c = 0 \quad \Rightarrow \quad$ point circle

$g^2 + f^2 - c < 0 \quad \Rightarrow \quad$ imaginary circle, with real centre, that is $(-g, -f)$

Note that every second degree equation in x & y, in which coefficient of x^2 is equal to coefficient of y^2 & the coefficient of xy is zero, always represents a circle.

(d) The equation of circle with (x_1, y_1) & (x_2, y_2) as extremities of its diameter is :

$$\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.}$$

Note that this will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).

2. INTERCEPTS MADE BY A CIRCLE ON THE AXES

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co-ordinate axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively. If :

$g^2 - c > 0 \quad \Rightarrow \quad$ circle cuts the x axis at two distinct points.

$g^2 - c = 0 \quad \Rightarrow \quad$ circle touches the x–axis

$g^2 - c < 0 \quad \Rightarrow \quad$ circle lies completely above or below the x–axis.

3. PARAMETRIC EQUATIONS OF A CIRCLE

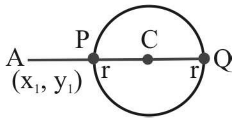
The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are :

$$x = h + r\cos\theta\ ;\quad y = k + r\sin\theta\ ;\ -\pi < \theta \le \pi$$

where (h, k) is the centre, r is the radius & θ is a parameter.

4. POSITION OF A POINT WITH RESPECT TO A CIRCLE

The point (x_1, y_1) is inside, on or outside the circle $S \equiv x_1^2 + y_1^2 + 2gx + 2fy + c = 0$.



according as $\mathbf{S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < = \text{ or } > 0.}$

Note...

The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.

5. LINE AND A CIRCLE

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r \iff$ the line does not meet the circle i.e. passes outside the circle.

(ii) $p = r \iff$ the line touches the circle. (It is tangent to the circle)

(iii) $p < r \iff$ the line is a secant of the circle.

(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then.

(i) $c^2 < a^2(1 + m^2) \iff$ the line is a secant of the circle.

(ii) $c^2 = a^2(1 + m^2) \iff$ the line touches the circle. (It is tangent to the circle)

(iii) $c^2 > a^2(1 + m^2) \iff$ the line does not meet the circle i.e. passes out side the circle.

6. TANGENT

(a) Slope form :

y = mx + c is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$.

Hence, equation of tangent is $\mathbf{y = mx \pm a\sqrt{1+m^2}}$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$, where $c = \pm a\sqrt{1+m^2}$.

(b) Point form :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $\mathbf{x x_1 + y y_1 = a^2}$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is :

$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$;

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Note...

In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{x_1y+xy_1}{2}$ and c remains as c.

(c) Parametric form :

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is : $\mathbf{x\cos\alpha + y\sin\alpha = a}$.

Note...

The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is :

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

7. FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is : $\mathbf{S_1 + K\,S_2 = 0}$

($K \neq -1$, provided the co-efficient of x^2 & y^2 in S_1 & S_2 are same)

(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by **S + KL = 0.**

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0}$$

where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$ where K is a parameter.

Note...

★ Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided co-efficient of xy = 0 and co-efficient of x^2 = co-efficient of y^2.

★ Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $\mu L_1L_3 + \lambda L_2L_4 = 0$ where value of μ and λ can be found out by using condition that co-efficient of x^2 = co-efficient of y^2 and co-efficient of xy = 0.

8. NORMAL

If a line is normal /orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is ;

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

9. PAIR OF TANGENTS FROM A POINT

The equation of a pair of tangents drawn from the point A (x_1, y_1) to the circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ is : $\mathbf{SS_1 = T^2}$

10. LENGTH OF A TANGENT AND POWER OF A POINT

The length of a tangent from an external point (x_1, y_1) to the circle :

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$\mathbf{L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}}$$

Square of length of the tangent from the point P is also called the power of point w.r.t. a circle. Power of a point w.r.t a circle remains constant.

Power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

11. DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{\ }$ times the original circle.

12. CHORD OF CONTACT

If two tangents PT_1 & PT_2 are drawn from the point P (x_1, y_1) to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

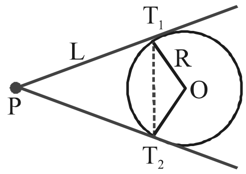
$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$ $T = 0$

Note...

Here R = radius, L = Length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.

(b) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R \ + L}}$



(c) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\dfrac{2RL}{L^2 - R}\right)$

(e) Equation of the circle circumscribing the triangle $PT_1\ T_2$ is :

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0.$

13. POLE AND POLAR (NOT IN SYLLABUS)

(i) If through a point P in the plane of the circle there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q and R is called the Polar of the point P; also P is called the Pole of the Polar.

(ii) The equation of the polar of a point P $(x_1\ y_1)$ w.r.t. the circle $x^2 + y^2 = a^2$ is given by $xx_1 + yy_1 = a^2$ and if the circle is general then the equation of the polar becomes $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ i.e. $T = 0$. Note that if the point $(x_1\ y_1)$ be on the circle then the tangent & polar will be represented by the same equation. Similary if the point $(x_1\ y_1)$ be outside the circle then the chord of contact and polar will be represented by the same equation.

(iii) Pole of a given line $Ax + By + C = 0$ w.r.t. circle $x^2 + y^2 = a^2$ is $\left(-\dfrac{Aa}{C}, -\dfrac{Ba}{C}\right)$

(iv) If the polar of a point P pass through a point Q then the polar of Q passes through P.

(v) Two lines L_1 & L_2 are conjugate of each other if Pole of L_1 lies on L_2 and vice versa. Similarly two points P and Q are said to be conjuagate of each other if the polar of P passes through Q and vice-versa.

14. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $\mathbf{T = S_1}$.

Note...

(i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.

(ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

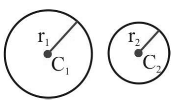
15. EQUATION OF THE CHORD JOINING TWO POINTS OF CIRCLE

The equation of chord PQ of the circle $x^2 + y^2 = a^2$ joining two points $P(\alpha)$ and $P(\beta)$ on it is given by. The equation of a straight line joining two point α and β on the circle $x^2 + y^2 = a^2$ is

$$\mathbf{x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}}$$

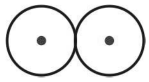
16. COMMON TANGENTS TO TWO CIRCLES

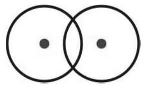
CASE | **NUMBER OF TANGENTS** | **CONDITION**

(i) 

4 common tangents $r_1 + r_2 < c_1c_2$.

(2 direct and 2 transverse)

(ii) 
3 common tangents $r_1 + r_2 = c_1c_2$.

(iii) 
2 common tangents $|r_1 - r_2| < c_1c_2 < r_1 + r_2$

(iv) 
1 common tangents $|r_1 - r_2| = c_1c_2$.

(v) 
No common tangent. $c_1c_2 < |r_1 - r_2|$

(Here c_1c_2 is distance between centres of two circles.)

Note...

(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

(ii) Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by : $L_{ext.} = \sqrt{d^2 - (r_1 - r_2)^2}$

and $L_{int.} = \sqrt{d^2 - (r_1 + r_2)^2}$, where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

17. ORTHOGONALITY OF TWO CIRCLES

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is : $\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$.

Note...

(a) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles. (Not in syllabus).

(c) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

18. RADICAL AXIS AND RADICAL CENTRE

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given by :

$S_1 - S_2 = 0$

i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$

The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

Note...

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

(d) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(e) Radical axis bisects a common tangent between the two circles.

(f) A system of circles, every two of which have the same radical axis, is called a coaxal system.

(g) Pairs of circles which do not have radical axis are concentric.