Chapter 5 : Continuity and Differentiability (XII)

Continuity

A function f is said to be continuous at a point $x = c$ if the following condition are satisfied :

(i) The function is defined at $x = c$ i.e. $f(c)$ is defined.

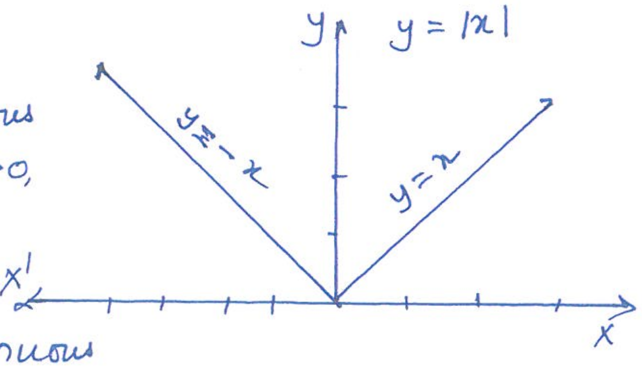
(ii) Limit of the function at $x = c$ exists i.e. $\lim_{x \to c} f(x)$ exists.

(iii) The value of function at $x = c$ equals the limit of the function at $x = c$ i.e. $\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions in this definition fail to hold, then f is called discontinuous at c and c is called a point of discontinuity of f.

If f is continuous at all points of an open intervals (a, b), then f is said to be continuous on (a, b). A function that is continuous on $(-\infty, \infty)$ is said to be continuous everywhere or simply continuous.

for examples

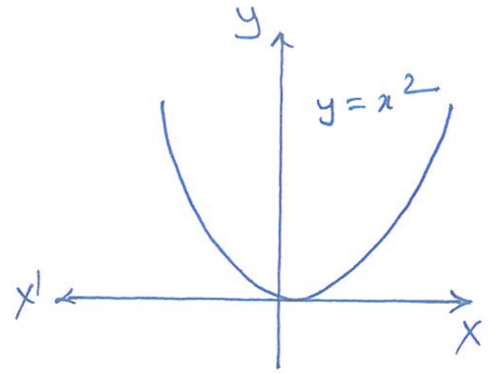


1) The function $f(x) = |x|$ is continuous at every value of x. for $x > 0$, we have $f(x) = x$, a polynomial.
for $x < 0$, we have $f(x) = -x$, another polynomial. So f is continuous for all $x \neq 0$.
Finally at origin $\lim_{x \to 0} |x| = 0 = f(0)$ i.e. $f(x) = f(0)$, so f is continuous at 0.
Hence, the function $f(x) = |x|$ is continuous at every value of x.

2) Let $f(x) = x^2$, then for any real number c.

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 = c^2 = f(c)$$

Hence, the function $f(x) = x^2$ is continuous for all value of x.



Note: we have earlier seen that $\lim_{x \to c} f(x)$ may exist without the function being defined at $x = c$. However, for the continuity of a function at $x = c$, the function must be defined at $x = c$.

3) A function $f(x)$ is defined as

$$f(x) = \begin{cases} \dfrac{x}{\sin 3x}, & \text{when } x \neq 0 \\ 3 & \text{when } x = 0 \end{cases}$$

is discontinuous at $x = 0$.

Sol^n: We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{x}{\sin 3x} = \lim_{x \to 0} \dfrac{3 \cdot x}{\sin 3x} \cdot \dfrac{1}{3}$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3}$$

$f(0) = 3$. Since, $\lim_{x \to 0} f(x) \neq f(0)$, therefore, $f(x)$ is not continuous at $x = 0$ ie. $x = 0$ is a point of discontinuity of $f(x)$.

(4) A function $f(x)$ is defined as $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3} & x \neq 3 \\ 5 & \text{if } x = 3. \end{cases}$ is continuous at 3.

We have $\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \dfrac{(x-3)(x+2)}{x - 3}$

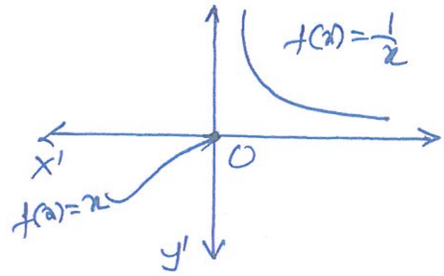
$$= \lim_{x \to 3} (x + 2) = 5.$$

$f(3) = 5 \Rightarrow f$ is continuous at 3.

4) Consider the function : $f(x) = \begin{cases} \frac{1}{x}, & x>0 \\ x, & x \le 0 \end{cases}$

For continuity, we should have :

$$\lim_{x \to 0^+} f(x) = 0 = f(0).$$



But $\lim_{x \to 0+} f(x)$ does not exist.

Hence, the function 'f' is discontinuous at $x=0$.

5) Let $f(x) = \dfrac{x^2-1}{x-1}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2$$

$f(1)$ is not defined.

$\therefore$ $\lim_{x \to 1} f(x) \neq f(1)$, so f is discontinuous at $x=1$.

6) Consider $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

From above, $\lim_{x \to 1} f(x) = 2 \neq f(1)$

$\therefore$ $f(x)$ is not continuous at $x=1$.

Kinds of Discontinuity

(1) Removable discontinuity :

If $x = c$ is a point of discontinuity of f such that $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$, then f is said to have removable discontinuity at $x = c$.

The discontinuity at 'c' can be removed by defining $f(c)$ in such a way that $\lim_{x \to c} f(x) = f(c)$.

Such a discontinuity is called Removable discontinuity.

(2) Discontinuity of first kind :

If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exists finitely but

$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then $f(x)$ is said to posses

Discontinuity of first kind at 'c'.

(3) Discontinuity of second kind :

If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ does not exist, then

$f(x)$ is said to posses Discontinuity of second kind at 'c'.

Example : $f(x) = \begin{cases} 1 + x^2, & 0 \le x \le 1 \\ 1 - x, & x > 0 \end{cases}$ at the point $x = 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = \lim_{h \to 0} [1 - (1 + h)] = \lim_{h \to 0} (-h) = 0$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 + x^2) = \lim_{h \to 0} [1 + (1 - h)^2] = \lim_{h \to 0} (h^2 - 2h + 2) = 2$$

$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$ $\qquad$ $\lim_{x \to 1} f(x)$ does not exist.

Hence, f is discontinuous at $x = 1$ and discontinuity of first kind.

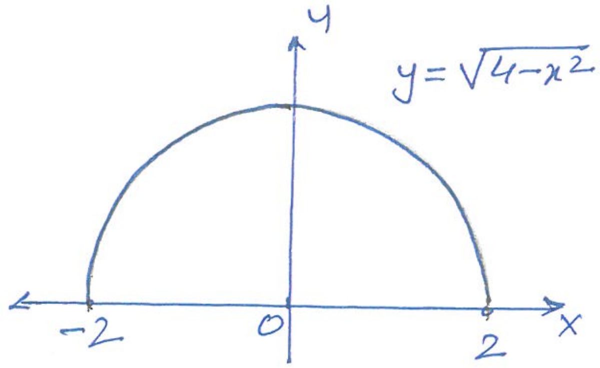
Continuity in an interval

(i) A function $f(x)$ is said to be continuous in an open interval (a,b) if it is continuous at every point in (a,b)

(ii) A function f is said to be continuous in a closed interval $[a,b]$ if it is continuous at every point of the open interval (a,b) and is continuous at the point 'a' from the right and continuous at b from left.

i.e. $\lim_{x \to a+} f(x) = f(a)$

$\lim_{x \to b-} f(x) = f(b)$.



For example, $f(x) = \sqrt{4-x^2}$

is continuous in $[-2, 2]$.

Continuity of Some Particular Function

Let $f(x) = x^2 - 3x + 2$, then $\lim_{x \to a} f(x) = a^2 - 3a + 2 = f(a) \; \forall \, a \in R$.

$\therefore$ $f(x)$ is continuous at every point in R.

$\Rightarrow$ $f(x)$ is continuous everywhere.

In fact, every polynomial function is continuous everywhere.

A constant function is a polynomial of degree zero. Hence, it is continuous everywhere.

(i) The function $\sin x$, $\cos x$, e^x, a^x are continuous everywhere.

(ii) The function $\sin^{-1} x$ and $\cos^{-1} x$ are continuous in $[-1, 1]$

(iii) The function $\operatorname{cosec}^{-1} x$ and $\sec^{-1} x$ are continuous in $(-\infty, -1] \cup [1, \infty)$.

Algebra of Continuous Functions

Theorem : Let 'f' and 'g' be real valued continuous functions defined in a neighbourhood of a point c. Then :

(i) αf is continuous at 'c' for all $\alpha \in R$.

(ii) $f+g$ and $f-g$ are continuous at 'c'

(iii) fg is continuous at 'c'

(iv) $\frac{f}{g}$ is continuous at 'c', provided $g(c) \neq 0$.

Note: If $f+g$ is continuous at 'c', it does not follow that both f and g are continuous at 'c'. Similar are the cases with $f-g$, fg and $\frac{f}{g}$.

Theorem : A polynomial function is continuous at each point of its domain.

Theorem : A rational function $\frac{g(x)}{h(x)}$ is continuous at each point of its domain.

Continuity of Composite functions

Composite of function, $(gof)(x) = g(f(x))$

Theorem : Let 'f' and 'g' be real functions such that 'fog' is defined. If 'g' is continuous at a point 'c' and if 'f' is continuous at $g(c)$, then (fog) is continuous at 'c'.

Solved Examples

1) Prove that the function $f(x) = 5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Solⁿ: (i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} 5x-3 = -3$

$f(x) = 5x-3, \quad f(0) = -3$

$\therefore \lim_{x\to 0} f(x) = f(0)$. Hence, f is continuous at $x=0$.

2) Examine the continuity of the function $f(x) = 2x^2-1$ at $x=3$.

Solⁿ: Let $f(x) = 2x^2-1$

$\lim_{x\to 3} f(x) = \lim_{x\to 3} 2x^2-1 = 2(3)^2-1 = 2\times 9-1 = 17$

$f(3) = 2\times 3^2-1 = 17 \qquad \therefore \lim_{x\to 3} f(x) = f(3)$

Hence, function is continuous.

3) Examine the following functions for continuity:

(i) $f(x) = x-5$ (ii) $f(x) = \frac{1}{x-5}$ (iii) $f(x) = \frac{x^2-25}{x+5}$

(iv) $f(x) = |x-5|$.

Solⁿ: (i) $f(x) = x-5$, is a polynomial function and therefore, it is continuous at every point.

(ii) $f(x) = \frac{1}{x-5}$ is not defined at $x=5$. So $f(x)$ is not continuous at $x=5$.

Let c be any arbitrary real number.

$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x-5} = \frac{1}{c-5} \qquad f(c) = \frac{1}{c-5} \qquad \therefore \lim_{x\to c} f(x) = f(c)$

Hence, f is continuous at every real number except at $x=5$.

(iii) $f(x) = \frac{x^2-25}{x+5}$

$f(x)$ is not defined at point $x=-5$.
Hence, domain of the given function $f(x)$ is $R-\{-5\}$

Let c be any arbitrary real number other than -5.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2-25}{x+5} = \frac{c^2-25}{c+5} = f(c)$.
$c \neq -5$.

$\therefore$ $f(x)$ is continuous at each real number c other than -5.

(iv) $f(x) = |x-5| = \begin{cases} 5-x & \text{if } x<5 \\ 0 & \text{if } x=5 \\ x-5 & \text{if } x>5. \end{cases}$

Since $f(x) = 5-x$ in $(-\infty, 5)$ is a polynomial function, hence $f(x)$ is continuous in $(-\infty, 5)$.

at $x=5$ $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (5-x) = 5-5 = 0$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x-5) = 5-5 = 0$.

$\therefore$ $\lim_{x \to 5^-} f(x) = f(5) = \lim_{x \to 5^+} f(x)$

$\therefore$ $f(x)$ is continuous at $x=5$.

Also, $f(x) = x-5$ in $(5, \infty)$ is a polynomial function and is continuous in $(5, \infty)$.

Hence, $f(x) = |x-5|$ is continuous at every real number.

4) Prove that the function $f(x) = x^n$ is continuous at $x=n$, where n is positive integer.

Soln: $f(x) = x^n$ is a polynomial function. Therefore it is continuous at $x=n$. $x \in R$.

5) Is the function f defined by $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 5 & \text{if } x > 1 \end{cases}$

continuous at $x=0$? At $x=1$? At $x=2$?

Soln: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} x = \lim\limits_{h\to 0} (0-h) = 0.$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} x = \lim\limits_{h\to 0} (0+h) = 0.$

$\therefore \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x) = 0.$

Also $f(0) = 0.$

$\therefore \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x) = f(0).$

Hence, f is continuous at $x=0$.

(ii) $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1-} x = \lim\limits_{h\to 0} (1-h) = 1$

$\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1+} 5 = 5.$

Hence $\lim\limits_{x\to 1+} f(x) \neq \lim\limits_{x\to 1-} f(x)$

Hence, f is not continuous at $x=1$.

(iii) at $x=2$, $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} 5 = 5.$ Also $f(2) = 5$.

$\therefore \lim\limits_{x\to 2} f(x) = f(2) = 5$

Hence, f is continuous at $x=2$.

(6) Show that the function $f(x) = 2x - |x|$ is continuous at $x=0$.

Sol^n $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$

$$\therefore \quad f(x) = 2x - |x| = \begin{cases} 2x - x & \text{if } x \geq 0 \\ 2x - (-x) & \text{if } x < 0 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 3x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} 3x = \lim_{h \to 0} 3(0-h) = 3(0-0) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = \lim_{h \to 0} (0+h) = 0.$$

Also $f(0) = 0$.

Hence, $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} 3x = f(0)$.

Hence, f is continuous at $x=0$.

7) For what value of k is the function

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ k & \text{if } x = 3 \end{cases}$$ continuous at $x=3$?

Sol^n: $\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{x^2-9}{x-3} = \lim_{x \to 3} \dfrac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} (x+3) = 6$

$\because f(x)$ is continuous at $x=3$, we have $\lim_{x \to 3} f(x) = f(3)$

$f(3) = k$

$$\therefore \lim_{x \to 3} f(x) = f(3) = k = 6$$

$\therefore$ $\underline{k = 6}$

(8) Show that the function

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{where } x \neq 0. \\ 2, & \text{where } x = 0. \end{cases}$$ is continuous at $x=0$.

Soln: $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin x}{x} + \cos x = \lim\limits_{x \to 0} \frac{\sin x}{x} + \lim\limits_{x \to 0} \cos x$

$= 1 + \cos 0^{\circ} = 1+1 = 2$

$f(0) = 2$ [Given].

$\therefore \lim\limits_{x \to 0} f(x) = f(0) = 2$

Hence, $f(x)$ is continuous at $x=0$.

9) Find the value of k so that

$$f(x) = \begin{cases} \frac{\sin kx}{x} & \text{for } x \neq 0 \\ 4+x & \text{for } x = 0 \end{cases}$$ is continuous at $x=0$.

Soln: $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin kx}{x} = \lim\limits_{x \to 0} \frac{\sin kx}{kx} \times k$

$= 1 \times k = k.$

$f(0) = 4+0 = 4.$

Now, f is continuous at $x=0$

$\therefore \lim\limits_{x \to 0} f(x) = f(0) = 4 = k$

$\therefore k = 4.$

10) For what value of k is the function

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$ continuous at $x=0$?

Soln: $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \times 2 = 1 \times 2 = 2$

$f(0) = k$ (given)

Since f is continuous at $x=0$,

$\lim_{x \to 0} f(x) = f(0) = k = 2$

$\therefore$ $k = 2$.

11) For what value of k is the function

$$f(x) = \begin{cases} \frac{\tan 2x}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$ [$k=2$]

12) Find the value of the constant k so that the function given below is continuous at $x=0$.

$$f(x) = \begin{cases} \frac{1-\cos 4x}{8x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$ [$k=1$]

13) A function f is defined by $f(x) = \begin{cases} \frac{1-\cos x}{x^2} & x \neq 0 \\ A & x = 0. \end{cases}$
Find A so that f is continuous at $x=0$

[$A = \frac{1}{2}$]

(14) Find all the points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ [f is discontinuous].

Chapter 5: Continuity and Differentiability (XII)

Differentiability

Introduction : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called the derivatives with respect to x of the function f.

The following theorem shows that a function must be continuous at each point where it is differentiable.

Theorem : If a function f is differentiable at a point c, then f is also continuous at c.

The converse, however, is false. A function may be continuous at a point but not differentiable there.

The derivatives of a function f is defined at those points where the limit in $f'(x)$ exists. If c is such a point, then we say that f is differentiable at c or f has a derivative at c.

Let $y = f(x)$ be any given function.

The derivatives of the function w.r.t x $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. provided the limit exists.

The function is said to be differentiable at $x = a$ if the right hand derivative (RHD) of a function f at a point a denoted by $Rf'(a)$, is defined as

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ (provided the limit exists.)}$$

where $h > 0$.

The left hand derivative (LHD) of a function f at a point a, denoted by $Lf'(a)$ is defined by

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} \text{ (provided the limit exists)}$$

where $h > 0$.

<u>Differentiability at a point :</u>

A function $f(x)$ is said to be differentiable at a point 'a' if

(i) both $Rf'(a)$ and $Lf'(a)$ exists and finite

(ii) $Rf'(a) = Lf'(a)$.

i.e. f is differentiable at a point a of its domain if and only if both $Rf'(a)$ and $Lf'(a)$ exist and are equal and their common value is the derivative $f'(a)$ of f at a.

Remember : If $Rf'(a)$ or $Lf'(a)$ does not exist or if both $Rf'(a)$ and $Lf'(a)$ exist but are not equal, then f is not differentiable at a.

<u>Theorem :</u> If a function f is differentiable at a point c, then f is also continuous at c.

Since f is differentiable at c i.e. $f'(c)$ exists,

we have $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Solved Examples

1) Show that the function $f(x) = |x-3|$ is not differentiable at $x = 3$.

Sol^n. $f(x) = |x-3|$

$$RHD = Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad h>0$$

$$Rf'(a) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0} \frac{|3+h-3| - 0}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1$$

$$Rf'(a) = 1.$$

$$LHD = Lf'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} \qquad h>0$$

$$= \lim_{h\to 0} \frac{f(3-h)-f(3)}{-h}$$

$$= \lim_{h\to 0} \frac{|3-h-3| - 0}{-h} = \lim_{h\to 0} \frac{|-h|}{-h}$$

$$= \lim_{h\to 0} (-1)$$

$$= -1$$

As $Rf(3) \neq Lf'(3)$, so $f(x)$ is not differentiable.

2) Examine the differentiability of the following at $x=0$

(i) $|x^2|$ (ii) $[x]$

(i) $$LHD = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{|x^2|-0}{x}$$

$$= \lim_{h\to 0-} \frac{|(0-h)^2|-0}{0-h} \qquad \text{Put } x = 0-h,\ h>0$$

$$= \lim_{h\to 0} \frac{h^2}{-h} = \lim_{h\to 0} (-h) = 0.$$

$$RHD = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - h} = \lim_{x \to 0+} \frac{|x^2| - 0}{x}$$

$$= \lim_{h \to 0} \frac{|(0+h)^2| - 0}{0+h} \qquad \text{putting } x = 0+h, \ h > 0$$

$$= \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0.$$

Since $LHD = RHD$ $\therefore$ $|x^2|$ is differentiable at $x = 0$.

(ii) $f(x) = [x]$

$$LHD = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{[x] - 0}{x}$$

$$= \lim_{h \to 0} \frac{[0-h] - 0}{0-h} = \lim_{h \to 0} \frac{[-h]}{-h} = \lim_{h \to 0} \frac{+1}{-h}$$

which does not exist.

Similarly, RHD does not exist.

Hence $[x]$ is not differentiable at $x = 0$.

(3) Examine the differentiability of the following function at $x = 0$:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Sol^n: $$LHD = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0-} \frac{x^2 \sin \frac{1}{x} - 0}{x}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin \frac{1}{0-h}}{0-h} \qquad \text{putting } x = 0-h, \ h > 0$$

$$LHD = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{-h}}{-h} = \lim_{h\to 0} +h \sin\frac{1}{h}$$ $\because (-h)^2 = h^2$, $\sin(-\theta) = -\sin\theta$.

$$= +\lim_{h\to 0} \frac{\sin\frac{1}{h}}{1/h} = +1$$

$LHD = +1$

$$RHD = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x-0}$$

$$= \lim_{h\to 0} \frac{(0+h)^2 \sin\frac{1}{0+h}}{0+h}$$ putting $x = 0+h$

$$= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h\cdot \sin\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\sin\frac{1}{h}}{1/h} = 1$$

$\therefore$ LHD = RHD, so $f(x)$ is differentiable at $x=0$.

(4) Show that the function f is defined as follows, is continuous at $x=2$ but not differentiable at $x=2$.

$$f(x) = \begin{cases} 3x-2 & 0 \le x \le 1 \\ 2x^2 - x & 1 < x \le 2 \\ 5x-4 & x > 2 \end{cases}$$

Soln: For continuity of $f(x)$ at $x=2$

$$LHL = \lim_{x\to 2-} f(x) = \lim_{x\to 2-} (2x^2 - x)$$

$$= \lim_{h\to 0} [2(2-h)^2 - (2-h)]$$ putting $x = 2-h$, $h>0$.

$$LHL = \lim_{h \to 0} \left[2(4+h^2-4h) - 2+h \right]$$

$$= \lim_{h \to 0} (8+2h^2-8h-2+h)$$

$$= \lim_{h \to 0} (2h^2-7h+6)$$

$$LHL = 6$$

$$RHL = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x-4) = \lim_{h \to 0} \left[5(2+h)-4 \right]$$ [putting $x=2+h$]

$$= \lim_{h \to 0} (10+5h-4] = \lim_{h \to 0} (5h+6) = 6.$$

$LHL = RHL$, So $f(x)$ is continuous at $x=2$.

<u>Differentiability at $x=2$</u>

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0} \frac{\{5(2+h)-4\} - \{2(2)^2-2\}}{h}$$

$h > 0.$

$$Rf'(2) = \lim_{h \to 0} \frac{(10+5h-4) - (8-2)}{h} = \lim_{h \to 0} \frac{(5h+6-6)}{h}$$

$$= \lim_{h \to 0} \frac{5h}{h} = \lim_{h \to 0} 5 = 5.$$

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h \to 0} \frac{\{2(2-h)^2-(2-h)\} - \{2(2)^2-2\}}{-h}$$

$h > 0$

$$= \lim_{h \to 0} \frac{\{2(4+h^2-4h)-2+h\} - \{8-2\}}{-h}$$

$$= \lim_{h \to 0} \frac{\{8+2h^2-8h-2+h\} - 6}{-h} = \lim_{h \to 0} \frac{(2h^2-7h+6-6)}{h}$$

$$= \lim_{h \to 0} \frac{h(2h-7)}{-h} = \lim_{h \to 0} (7-2h) = 7$$

$Rf'(2) \neq Lf'(2)$

(20) Not differentiable at $x=2$.

(5) Show that $f(x) = |x|$ is not differentiable at $x=0$.

Sol^n: $RHD = Rf'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ $\quad h\geq 0$

$$RHD = Rf'(0) = \lim_{h\to 0} \frac{|0+h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$LHD = Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|0-h|-0}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-1} = -1$$

As $LHD \neq RHD$, so $f(x) = |x|$ is not differentiable at $x=0$.

(6) Prove that $f(x) = x|x|$ is differentiable for all real values of x.

Sol^n: $f(x) = x|x|$ $\qquad f(x) = \begin{cases} x^2, & \text{if } x\geq 0 \\ -x^2, & \text{if } x \leq 0 \end{cases}$

$$LHD = Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-(0-h)^2-0}{-h}$$

$$= \lim_{h\to 0} \frac{\cancel{-}h^2}{\cancel{-}h} = \lim_{h\to 0} h = 0.$$

$$RHD = Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to} \frac{(0+h)^2-0}{h} = \lim_{h\to 0} \frac{h^2}{h}$$

$$= \lim_{h\to 0} h = 0$$

$$Lf'(0) = Rf'(0)$$

Hence, $f(x)$ is differentiable for real values of x.

7) Prove that the function $f(x) = 1 + |\sin x|$ is not differentiable at $x = 0$.

Sol^n: $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{h} = \lim_{h \to 0} \frac{\cancel{1} + |\sin(0-h)| - \cancel{1}}{-h}$$

$$= \lim_{h \to 0} \frac{|\sin(-h)|}{-h} = \lim_{h \to} \frac{|-\sin h|}{-h} = -1$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\cancel{1} + |\sin(0+h)| - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$Lf'(0) \neq Rf'(0)$ Hence $f(x)$ is not differentiable at $x = 0$.

8) If $f(x) = \sin|x|$, then prove that $f(x)$ is not differentiable at $x = 0$.

Sol^n: $f(x) = \sin|x|$ $$f(x) = \begin{cases} \sin x, & x > 0 \\ 0, & x = 0 \\ -\sin x, & x < 0 \end{cases}$$

$$f(0) = 0$$

$$LHD = Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-\sin(0-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-(-\sin h)}{-h} = -\lim_{h \to 0} \frac{\sin h}{h} = -1$$

$$RHD = Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$LHD \neq RHD$, Hence $f(x)$ is not differential at $x = 0$.

9) $f(x) = \begin{cases} 2a - x & \text{in } -a < x < a \\ 3x - 2a & \text{in } a \leq x \end{cases}$

Prove that $f(x)$ is not differentiable at $x = a$.

Soln at $x = a$ $\quad LHD = \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

$$LHD = \lim_{h \to 0} \frac{2a - (a-h) - (2a-a)}{-h}$$

$$= \lim_{h \to 0} \frac{2a - a + h - a}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$$RHD = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h) - 2a - (3a - 2a)}{h}$$

$$= \lim_{h \to 0} \frac{3a + 3h - 2a - a}{h} = \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}} = 3$$

$LHD \neq RHD$. Hence, $f(x)$ is not differentiable at $x = a$.

10) Is $f(x) = |x-1| + |x-2|$ differentiable at $x = 2$?

Soln: $\quad f(x) = |x-1| + |x-2|$

at $x = 2$, $\quad f(2) = |2-1| + |2-2| = 1 + 0 = 1$

$$LHD = Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$f(2-h) = |2-h-1| + |\cancel{2}-h-\cancel{2}|$$
$$= |1-h| + |-h|$$
$$= 1 - \cancel{h} + \cancel{h}$$
$$f(2-h) = 1$$

$$\therefore Lf'(2) = \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} \frac{0}{-h} = 0.$$

at $x=2$ RHD $= Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

$f(2) = 1$

$f(2+h) = |2+h-1| + |2+h-2|$

$= |1+h| + |h|$

$= 1+h+h$

$= 1+2h$

$\therefore Rf'(2) = \lim_{h\to 0} \frac{\cancel{1}+2h-\cancel{1}}{h} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}} = 2$

LHD $\neq$ RHD. Hence, $f(x)$ is not derivable at $x=2$.

11) Show that the function f defined as follows is continuous, but not differentiable at $x=2$.

$$f(x) = \begin{cases} 1+x, & \text{when } x \leq 2 \\ 5-x, & \text{when } x > 2 \end{cases}$$

Soln:

RHL $= \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \{5-(2+h)\} = \lim_{h\to 0} (5-2-h)$

$= \lim_{h\to 0} (3-h) = 3$

LHL $= \lim_{h\to 0} f(2-h) = \lim_{h\to 0} 1+2-h = \lim_{h\to 0} 3-h = 3$

LHL = RHL. So $f(x)$ is continuous at $x=2$

$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{5-(2+h)-3}{h} = \lim_{h\to 0} \frac{3-h-3}{h} = -1$

$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{1+2-h-(1+2)}{-h} = \lim_{h\to 0} \frac{-h}{-h} = 1$

$Rf'(2) \neq Lf'(2)$. Hence $f(x)$ is not differentiable at $x=2$.

12) Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x=1$, $x=2$.

Differentiation

Let $y = f(x)$ be a function of x.
As x changes from x to $x+h$,
y changes from $f(x)$ to $f(x+h)$.

Differential coefficient of y w.r.t x, which is usually denoted by $\frac{dy}{dx}$ or $f'(x)$ symbolically is given by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exists

Basic Rules for finding Derivatives of functions

1) The derivative of a constant function is zero
If $f(x) = c$, then $f'(x) = 0$ $\frac{d}{dx}(c) = 0$.

2) If n is a rational number, then the derivatives of x^n is nx^{n-1}, where n is a fixed number.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

3) (i) $\frac{d}{dx}(\sin x) = \cos x$ (ii) $\frac{d}{dx}(\cos x) = -\sin x$ (iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

4) (i) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ (ii) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(iii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$.

5) (i) $\frac{d}{dx}(a^x) = a^x \log_e a$ (ii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ (iii) $\frac{d}{dx}(e^x) = e^x$

6) $\frac{d[f(x) \cdot g(x)]}{dx} = f(x) \frac{d[g(x)]}{dx} + g(x) \frac{d[f(x)]}{dx}$

7) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d[f(x)]}{dx} - f(x) \frac{d[g(x)]}{dx}}{[g(x)]^2}$$

Chain Rule

One of the most powerful differentiable rule - The chain Rule. This rules deals with composite functions.
The chain rule will enable us to differentiate complicated functions using known derivatives of simple functions.

Although we can differentiate $\sqrt{x}$ and x^2+1, we cannot yet differentiate $\sqrt{x^2+1}$. To do so, we need a rule that tells us how to find the derivatives of a composite functions.

A composite function is a function of another function

Example, $y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$.

may be written as $y = u^{1/2}$ where $u = x^2+1$.

Here y is a function of u and u is a function of x.
So, y is a function of x.

Theorem: The chain rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The following examples illustrate :

1) Differentiate the functions with respect to x:
(i) $\sin(x^2+5)$ (ii) $\cos(\sin x)$ (iii) $\sin(ax+b)$

Solⁿ: (i) We have $y = \sin(x^2+5)$ let $u = x^2+5$

then $y = \sin u$ and $u = x^2+5$

$\frac{dy}{du} = \cos u$ $\frac{du}{dx} = 2x$

using chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \times 2x = 2x \cos(x^2+5)$.

(ii) let $y = \cos(\sin x)$ let $u = \sin x$
then $y = \cos u$ where $u = \sin x$

$\frac{dy}{du} = -\sin u$ $\frac{du}{dx} = \cos x$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \times \cos x = -\cos x \cdot \sin(\sin x)$.

(iii) $y = \sin(ax+b)$ let $u = ax+b$

$y = \sin u$, $u = ax+b$

$\frac{dy}{du} = \cos u$ $\frac{du}{dx} = a$

$\frac{dy}{dx} \cdot \frac{dy}{du} \cdot \frac{du}{dx} = a \cos u = a \cos(ax+b)$.

2) Differentiate the following w.r.t to x
(i) $\cos\sqrt{x}$ (ii) $\sec(\tan\sqrt{x})$

Solved Examples

1) Differentiate w.r.t. to x

(i) $\cos^3 x$ (ii) $\sqrt{\sin x}$ (iii) $\sin 4x$ (iv) $\cos x^3$

(v) $\tan\sqrt{x}$ (vi) $\operatorname{cosec}(1+x^2)$ (vii) $\sec(\tan x)$

Ans (i) $-3\cos^2 x\cdot\sin x$ (ii) $\dfrac{\cos x}{2\sqrt{\sin x}}$ (iii) $4\cos 4x$

(iv) $-3x^2\sin x^3$ (v) $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (vi) $-2x\operatorname{cosec}(1+x^2)\cot(1+x^2)$

(vii) $\sec(\tan x)\cdot\tan(\tan x)\cdot\sec^2 x$

2) Differentiate w.r.t. x.

(i) $\sqrt{1+\tan x}$ (ii) $\sqrt{1+\cot x}$

Ans. (i) $\dfrac{\sec^2 x}{2\sqrt{1+\tan x}}$ (ii) $-\dfrac{\operatorname{cosec}^2 x}{2\sqrt{1+\cot x}}$

3) Differentiate w.r.t. x :

(i) $\sin(3x^{1/3})$, $x>0$ (ii) $[\sin(3x+2)]^{7/2}$ (iii) $\cos(1-x^2)^2$

(iv) $\cos(\sin x^2)$ (v) $\sqrt{\sin x^3}$ (vi) $\operatorname{cosec}(1+x^2)$ (vii) $\cos\sqrt{x^2+1}$

(viii) $\sec^2\dfrac{x}{a}$ (ix) $\tan^4(x^2)$ (x) $\sec^n\sqrt{x}$ (xi) $\sec\sqrt{1+x^2}$

Ans (i) $x^{-\frac{2}{3}}\cos(3x^{\frac{1}{3}})$ (ii) $\dfrac{21}{2}\cos(3x+2)[\sin(3x+2)]^{\frac{5}{2}}$

(iii) $4x(1-x^2)\sin(1-x^2)^2$ (iv) $-2x\cos x^2\sin(\sin x^2)$

(v) $\dfrac{3x^2\cos x^3}{2\sqrt{\sin x^3}}$ (vi) $-2x\operatorname{cosec}(1+x^2)\cot(1+x^2)$

(vii) $\dfrac{-x\sin\sqrt{x^2+1}}{\sqrt{x^2+1}}$ (viii) $\dfrac{2}{a}\sec^2\dfrac{x}{a}\tan\dfrac{x}{a}$ (ix) $8x\tan^3(x^2)\sec^2(x^2)$

(x) $\dfrac{n}{2\sqrt{x}}\sec^n(\sqrt{x})\tan(\sqrt{x})$ (xi) $\dfrac{x}{\sqrt{1+x^2}}\sec\sqrt{1+x^2}\tan\sqrt{1+x^2}$

4) Differentiate the following functions w.r.t. x

(i) $x^4 \sin 2x$ (ii) $x^2 \sin \frac{1}{x}$ (iii) $\sin^5 x \cdot \sin x^5$

(iv) $\sin x^2 \cdot \tan x^3$.

<u>Ans</u>: (i) $2x^3(x\cos 2x + 2\sin 2x)$ (ii) $-\cos\left(\frac{1}{x}\right) + 2x\sin\frac{1}{x}$

(iii) $5\sin^4 x\,(x^4 \sin x \cdot \cos x^5 + \cos x \times \sin x^5)$.

(iv) $x\,[3x \sin x^2 \cdot \sec^2 x^3 + 2\cos x^2 \tan x^3]$

5) Differentiate each of the following w.r.t. x

(i) $\cos(\sin\sqrt{ax+b})$ (ii) $\cos(\sin x^2)$ at $x = \frac{\pi}{2}$

(iii) $\sin\left(\frac{1+x^2}{1-x^2}\right)$ (iv) $\sqrt{\cos(1+x^2}$ (v) $\left(\tan\sqrt{1+x^2}\right)^{\frac{1}{2}}$

<u>Ans</u> (i) $-\frac{a}{2\sqrt{ax+b}} \cdot \sin[\sin\sqrt{ax+b}]\,\cos\sqrt{ax+b}\,]$

(ii) $0 = -2\sqrt{\frac{\pi}{2}} \cdot \cos\frac{\pi}{2}\,\sin\left(\sin\frac{\pi}{2}\right)$. (iii) $\frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$

(iv) $-\frac{x\sin(1+x^2)}{\sqrt{\cos(1+x^2)}}$ (v) $\frac{x}{2\sqrt{1+x^2}} \cdot \frac{\sec^2\sqrt{1+x^2}}{\sqrt{\tan\sqrt{1+x^2}}}$

6) If $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$, show that $\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$

7) If $y = \log\sqrt{\frac{1+x\cos x}{1-x\cos x}}$, find $\frac{dy}{dx}$ in simplified form

$\left[\text{Ans. } \frac{dy}{dx} = \frac{\cos x - x\sin x}{1 - x^2\cos^2 x}\right]$

8) If $y = (x + \sqrt{x^2+a^2})$ prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$

9) If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ $(x \neq y)$, then prove that

$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Solutions

1) (i) let $y = \cos^3 x = (\cos x)^3$ let $u = \cos x$

then $y = u^3$ $u = \cos x$

$\frac{dy}{du} = 3u^2 = 3\cos^2 x$ $\frac{du}{dx} = -\sin x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3\cos^2 x\,(-\sin x) = -3\cos^2 x \cdot \sin x$.

(ii) $y = \sqrt{\sin x}$ $y = \sqrt{u}$ $u = \sin x$

$\frac{dy}{du} = \frac{1}{2} u^{\frac{1}{2}-1} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\sin x}}$

$\frac{du}{dx} = \cos x$ $\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{\sin x}} \times \cos x$

$\therefore \frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x}}$

(iii) $y = \sin 4x$ $y = \sin u$, $u = 4x$

$\frac{dy}{du} = \cos u = \cos 4x$ $\frac{du}{dx} = 4$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4\cos 4x$.

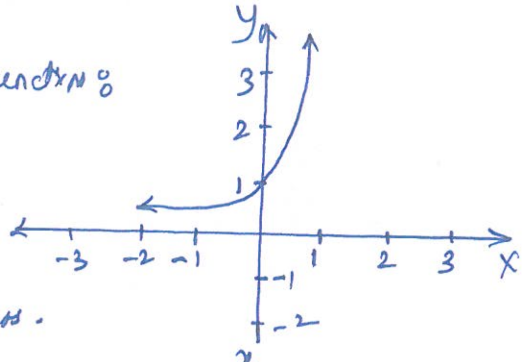
(iv) $y = \cos x^3$ $y = \cos u$, $u = x^3$

$\frac{dy}{du} = -\sin u = -\sin x^3$ $\frac{du}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -3x^2 \sin x^3$

Exponential and Logarithmic Functions.

An exponential function is a function of the form $f(x) = b^x$ where b is a positive real number and $b \neq 1$.



$y = 3^x$

Some salient features of the exponential function:

1) Domain of exponential function is R, the set of all real numbers.
2) Range of the exponential function is the set of all positive real numbers.
3) y-intercept is 1. The point $(0,1)$ is always on the graph of the exponential function.
4) Exponential function is ever increasing.
5. For large negative values of x, the exponential function is very close to 0.

The Base e:

The e is one of the most important numbers in mathematics.

The few digits are:

2.7182818284590452353602874... (and more).

e is an irrational number

e is the base of natural logarithms.

The value of e is equal to

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \text{ etc.}$$

It is often called Euler's number.

Value of e lies between 2 and 3.

The exponential function $f(x) = e^x$, whose base is the number e is usually referred to as natural exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

Logarithmic function to the base a

It is defined, where $a > 0$ and $a \neq 1$, by $y = \log_a x$

(read as "y is the logarithm of x to the base a")

$$y = \log_a x \quad \text{if} \quad x = a^y$$

If the base of the logarithmic function is the number e, then we have the natural logarithmic function.

$$x = e^y \qquad y = \log_e x \quad . \quad y = \ln x$$

Theorems:

(i) The derivative of e^x, $\frac{d(e^x)}{dx} = e^x$

(ii) The derivative of $\log x$, $\frac{d(\log_a x)}{dx} = \frac{1}{x} \log_a e$

if $a = e$ then $\frac{d(\log_e x)}{dx} = \frac{1}{x} \log_e e = \frac{1}{x}$

$\therefore \quad \frac{d(\log_a x)}{dx} = \frac{1}{x} \log_a e \quad \& \quad \frac{d(\log_e x)}{dx} = \frac{1}{x}$

Remember:

(i) $\log_a (mn) = \log_a m + \log_a n$

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

(iv) $\log_b N = \log_a N \times \log_b a$

Solved Examples

1) Differentiate the following w.r.t. x

(i) $\sin(\log x)$ (ii) $e^{\cos x}$

Sol^n: (i) $y = \sin(\log x)$

$y = \sin u$, $u = \log x$

$\frac{dy}{du} = \cos u$ $\frac{du}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{x} \cos(\log x)$

(ii) $y = e^{\cos x}$ $y = e^{u}$ $u = \cos x$

$\frac{dy}{du} = e^{u} = e^{\cos x}$ $\frac{du}{dx} = -\sin x$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin x \cdot e^{\cos x}$

2) Differentiate the following functions w.r.t to x

(i) $\log(e^{mx} + e^{-mx})$. (ii) $\log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

Sol^n: (i) $y = \log(e^{mx} + e^{-mx})$

$y = \log u$, $u = e^{mx} + e^{-mx}$

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{e^{mx}+e^{-mx}}$; $\frac{du}{dx} = \frac{d(e^{mx}+e^{-mx})}{dx}$

$= \frac{d(e^{mx})}{dx} + \frac{d(e^{-mx})}{dx}$

Let $t = mx$ $\frac{dz}{dt} = e^{t} = e^{mx}$

$z = e^{mx} = e^{t}$

$\frac{dt}{dx} = m$

$$\frac{dz}{dx} = e^{mx} \cdot m = me^{mx}$$

$$\therefore \frac{d(e^{mx})}{dx} + \frac{d(e^{-mx})}{dx} = me^{mx} - me^{-mx} = m(e^{mx} - e^{-mx})$$

$$\therefore \frac{dy}{dx} = \frac{m(e^{mx} - e^{-mx})}{e^{mx} + e^{-mx}}$$

(ii) $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$y = \log u$, $u = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\frac{dy}{du} = \frac{1}{u} = \frac{\sqrt{x}}{x+1}$$

$$\frac{du}{dx} = \frac{d(x^{\frac{1}{2}})}{dx} + \frac{d(x^{-\frac{1}{2}})}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}x^{-3/2}\right)$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} = \frac{x-1}{2x\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{x}}{x+1} \cdot \frac{(x-1)}{2x\sqrt{x}}$$

3) Differentiate following functions w.r.t. x :

(i) $(x^2 + \log x)^4$ (ii) $e^{2\log x + 3x}$

Solⁿ: $y = (x^2 + \log x)^4$

$y = u^4$ $u = x^2 + \log x$

$\frac{dy}{du} = 4u^3$ $\frac{du}{dx} = 2x + \frac{1}{x}$

$= 4(x^2 + \log x)^4$ $\frac{dy}{dx} = 4(x^2 + \log x)^4\left(2x + \frac{1}{x}\right)$

(ii) $y = e^{2\log x + 3x}$

$= e^{2\log x} \cdot e^{3x}$

$\log m^n = n \log m$ $\left[\because 2\log x = \log x^2\right]$

$\therefore y = e^{\log x^2} \cdot e^{3x}$

$y = x^2 \cdot e^{3x}$ $\left[\because e^{\log x^2} = x^2\right]$

$\frac{dy}{dx} = x^2 \frac{d}{dx}(e^{3x}) + e^{3x} \frac{d}{dx}(x^2)$

$= x^2 e^{3x} + e^{3x} \cdot 2x \cdot e^{3x}$

$\frac{dy}{dx} = x(2+3x)\, e^{3x}$

4) Differentiate the following function w.r.t. x :

(i) $\log \sin(e^x + 5x + 8)$ (ii) $\frac{x^2}{e^{1+x^2}}$

Ans. (i) $(e^x + 5)\cot(e^x + 5x + 8)$

(ii) $\frac{2x(1-x^2)}{e^{1+x^2}}$

5) Differentiate the following function w.r.t. x :

(i) $e^{2x}\sin 3x$ (ii) $e^{\sqrt{1-x^2}}\tan x$ (iii) $e^x \log(1+x^2)$

Ans (i) $(3\cos 3x + 2\sin 3x)\, e^{2x}$ (ii) $e^{\sqrt{1-x^2}}\left[\sec^2 x - \frac{x\tan x}{\sqrt{1-x^2}}\right]$

(iii) $e^x\left[\log(1+x^2) + \frac{2x}{1+x^2}\right]$

MISCELLANEOUS EXAMPLES

1) Differentiate the following :

(i) $\log \tan \frac{x}{2}$ (ii) $\log(\sec x + \tan x)$ (iii) $\sin(\log \cos x)$

ANS (i) $\operatorname{cosec} x$ (ii) $\sec x$ (iii) $-\tan x \cdot \cos(\log \cos x)$

2) Differentiate each of the following functions w.r.t. x :

(i) $\log(\cos x^2)$ (ii) $\cos(\log x)^2$ (iii) $\log\sqrt{\frac{1-\cos x}{1+\cos x}}$

ANS (i) $-2x \tan x^2$ (ii) $-\frac{2(\log x)\sin(\log x)^2}{x}$ (iii) $\operatorname{cosec} x$.

3) Find the derivative of $\log\left(\sin\sqrt{x^2+1}\right)$ ANS. $\frac{x\cot\sqrt{x^2+1}}{\sqrt{x^2+1}}$

4) Find the derivative of $\log\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)$ ANS. $\frac{1}{2}\sec\frac{x}{2}$

5) Find the derivatives of $\log\left(\sqrt{x+1}+\sqrt{x-1}\right)$ ANS. $\frac{1}{2\sqrt{x^2-1}}$

6) Find the derivatives of $\cos(\log \sin x)$ ANS. $-\cot x \sin(\log \sin x)$

7) Find the derivative of $x^3 \cdot e^x \log_e\sqrt{x}$

ANS. $3x^2 e^x \log_e\sqrt{x} + x^3 e^x \log_e\sqrt{x} + \frac{x^2}{2}e^x$

8) Differentiate : (i) $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ (ii) $\log\left(\sec\frac{x}{2}+\tan\frac{x}{2}\right)$

(iii) $\log[\log(\log x)]$ ANS. (i) $\sec x$ (ii) $\frac{1}{2}\sec\frac{x}{2}$ (iii) $\frac{1}{x\log x\log(\log x)}$

9) Differentiate : (i) $\log\sqrt{\frac{x-1}{x+1}}$ (ii) $\log\left(x+\sqrt{x^2+a^2}\right)$

ANS. (i) $\frac{1}{x^2-1}$ (ii) $\frac{1}{\sqrt{x^2+a^2}}$

Derivatives of Inverse Trigonometric functions.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

Remember:

(i) $\cos 2A = \cos^2 A - \sin^2 A$

$= \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(ii) $\sin 2A = 2\sin A \cdot \cos B$

$= \dfrac{2\tan A}{1+\tan^2 A}$

(iii) $2\sin^2 A = 1-\cos 2A$

(iv) $2\cos^2 A = 1+\cos 2A$

(v) $\tan\left(\frac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$

(vi) $\tan\left(\frac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A}$

(vii) $\sin 3A = 3\sin A - 4\sin^3 A$

(viii) $\cos 3A = 4\cos^3 A - 3\cos A$

(ix) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(ix) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(x) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

(xi) $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(xii) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(xiii) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(xvi) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{(1-x^2)(1-y^2)}\right\}$

(xv) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Things - To - Remember (Trigonometric functions - XI)

$1^\circ = 60'$ $\quad 1' = 60''$ $\quad \theta(\text{rad}) = \frac{l}{r}$ $\quad \pi(\text{rad}) = 180^\circ$

$\sin(2n\pi + x) = \sin x$ $\quad \cos(2n\pi + x) = \cos x$ $\quad n \in Z$

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

$\cos(-x) = \cos x$ $\quad \sin(-x) = -\sin x$ $\quad (a, b)$ $\quad a = \cos x$ $\quad b = \sin x$

$\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$ $\quad \cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ $\quad \sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$ $\quad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$\cot(A+B) = \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$ $\quad \cot(A-B) = \frac{\cot A \cdot \cot B - 1}{\cot B - \cot A}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$ $\quad \sin 2A = 2 \sin A \cdot \cos B = \frac{2 \tan A}{1 + \tan^2 A}$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ $\quad \sin 3A = 3 \sin A - 4 \sin^3 A$ $\quad \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$ $\quad \cos A - \cos B = -2 \sin \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$ $\quad \sin A - \sin B = 2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$ $\quad -2 \sin A \cdot \sin B = \cos(A+B) - \cos(A-B)$

$2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$ $\quad 2 \cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

$\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$ $\quad 2 \sin^2 A = 1 - \cos 2A$

$\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B)$ $\quad 2 \cos^2 A = 1 + \cos 2A$

$\tan(45^\circ + \theta) = \frac{1 + \tan\theta}{1 - \tan\theta}$

$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

Inverse Trigonometric function (XII)

(1) $\sin^{-1}(\sin x) = x \in [-1, 1]$

$\sin^{-1}(\sin\theta) = \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(2) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$

$\cos^{-1}\frac{1}{x} = \sec^{-1}x$

$\tan^{-1}\frac{1}{x} = \cot^{-1}x$

(3) $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

(4) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$

(5) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

(6) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(7) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

8) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

$\operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

9) $\sin^{-1}x \pm \sin^{-1}y$

$= \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

10) $\cos^{-1}x \pm \cos^{-1}y$

$= \cos^{-1}\left\{xy \mp \sqrt{(1-x^2)(1-y^2)}\right\}$

Solved Examples

1) Find the derivatives :

(i) $\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ (ii) $\cos^{-1}\sqrt{\frac{1+\cos x}{2}}$ (iii) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

(iv) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ (v) $\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$

Sol^n: (i) $y = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$1+\cos 2A = 2\cos^2 A$ $\sin 2A = 2\sin A \cdot \cos B$

$\therefore 1+\cos x = 2\cos^2\frac{x}{2}$ $\therefore \sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$$\therefore \frac{1+\cos x}{\sin x} = \frac{\cancel{2}\cos^{\cancel{2}}\frac{x}{2}}{\cancel{2}\cdot\sin\frac{x}{2}\cdot\cancel{\cos\frac{x}{2}}} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \cot\frac{x}{2}$$

$$\therefore y = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \qquad \therefore \frac{dy}{dx} = \frac{1}{2}$$

(ii) $$y = \cos^{-1}\sqrt{\frac{1+\cos x}{2}} = \cos^{-1}\sqrt{\frac{\cancel{2}\cos^2\frac{x}{2}}{\cancel{2}}} = \cos^{-1}\left(\cos\frac{x}{2}\right) = \frac{x}{2}$$

$$\therefore y = \frac{x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}.$$

(iii) $$y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\frac{x}{2}}{\cancel{2}\cos^2\frac{x}{2}}} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

$(1-\cos 2A = 2\sin^2 A)$

$$\therefore y = \frac{x}{2} \qquad \frac{dy}{dx} = \frac{1}{2}$$

(iv) $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

we know, $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2} = \frac{1}{1+\frac{1-x}{1+x}} = \frac{1+x}{1+\cancel{x}+1-\cancel{x}}$

$\frac{dy}{dx} = \frac{1+x}{2} = \frac{1}{2}(1+x)$

Since $x \in R$, this method is not correct

Because $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$, $x \in R$.

put $x = \cos\theta$ $\therefore \frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\cancel{2}\sin^2\frac{\theta}{2}}{\cancel{2}\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$

$\therefore y = \tan^{-1}\sqrt{\tan^2\frac{\theta}{2}} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1}x$

$\therefore \frac{dy}{dx} = \frac{1}{2} \times \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{2\sqrt{1-x^2}}$

(v) $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) = \tan^{-1}\left(\frac{\cancel{2}\sin^{\cancel{2}}\frac{x}{2}}{\cancel{2}\cancel{\sin\frac{x}{2}}\cdot\cos\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$

$\therefore y = \frac{x}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

2) Differentiate w.r.t. x :

(i) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (ii) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ (iii) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

(iv) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

Sol^n: $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

put $x = \tan\theta$

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta \quad \left[\because \cos 2\theta = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$

$$= \sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$\therefore y = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - 2\theta\right)\right) = \frac{\pi}{2} - 2\theta$$

$$\therefore y = \frac{\pi}{2} - 2\tan^{-1}x \quad \therefore \frac{dy}{dx} = 0 - 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-2}{1+x^2}$$

(ii) $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

$$\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1-\tan x}{1+\tan x} = \tan\left(\frac{\pi}{4} - x\right)$$

$$\therefore y = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) = \frac{\pi}{4} - x$$

$$\frac{dy}{dx} = -1.$$

(iii) $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

$\sin x = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}$ $\quad 1+\cos x = 2\cos^2\frac{x}{2}$

$\therefore \frac{\sin x}{1+\cos x} = \frac{\cancel{2}\cdot\sin\frac{x}{2}\cdot\cancel{\cos\frac{x}{2}}}{\cancel{2}\cos^2\frac{x}{2}} = \tan\frac{x}{2}$

$y = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{1}{2}$

(iv) $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$\frac{\cos x}{1+\sin x} = \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)} = \frac{\cancel{2}\cdot\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cancel{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}}{\cancel{2}\cdot\cos^{\cancel{2}}\left(\frac{\pi}{4}-\frac{x}{2}\right)}$

$= \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

$\therefore y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4}-\frac{x}{2}$

$\frac{dy}{dx} = -\frac{1}{2}$

3) Differentiate w.r.t. x :

(i) $\tan^{-1}\left(\frac{3-2x}{1+6x}\right)$ $\quad$ (ii) $\tan^{-1}\left(\frac{2x}{1-15x^2}\right)$

(iii) $\cot^{-1}\left(\frac{3-2\tan x}{2+3\tan x}\right)$ $\quad$ (iv) $\cot^{-1}\left(\frac{5+4x}{5x-4}\right)$

Soln: (i) We know $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$y = \tan^{-1}\left(\frac{3-2x}{1+3\cdot 2x}\right) = \tan^{-1}3 - \tan^{-1}2x$

$\frac{dy}{dx} = \frac{\tan(\tan^{-1}3)}{dx} - \frac{\tan(\tan^{-1}2x)}{dx} = 0 - \frac{1}{1+4x^2}\times\frac{d}{dx}(2x) = -\frac{2}{1+4x^2}$

(ii) $y = \tan^{-1}\left(\frac{2x}{1-15x^2}\right)$

$y = \tan^{-1}\left(\frac{5x - 3x}{1 - 5x\cdot 3x}\right) = \tan^{-1}5x - \tan^{-1}3x$

$\frac{dy}{dx} = \frac{d(\tan^{-1}5x)}{dx} - \frac{d(\tan^{-1}3x)}{dx}$

$= \frac{1}{1+25x^2} \times 5 - \frac{1}{1+9x^2} \times 3$

$= \frac{5}{1+25x^2} - \frac{3}{1+9x^2}$

(iii) $y = \cot^{-1}\left(\frac{3 - 2\tan x}{2 + 3\tan x}\right)$

$= \tan^{-1}\left(\frac{2 + 3\tan x}{3 - 2\tan x}\right)$

$= \tan^{-1}\left[\frac{\frac{2}{3} + \tan x}{1 - \frac{2}{3}\tan x}\right] = \tan^{-1}\frac{2}{3} + \tan^{-1}(\tan x)$

$y = \tan^{-1}\frac{2}{3} + x \qquad \therefore \frac{dy}{dx} = 1$

(iv) $y = \cot^{-1}\left(\frac{5+4x}{5x-4}\right) = \tan^{-1}\left(\frac{5x-4}{5+4x}\right) = \tan^{-1}\left(\frac{x - \frac{4}{5}}{1 + \frac{4}{5}x}\right)$

$y = \tan^{-1}x - \tan^{-1}\frac{4}{5}$

$\frac{dy}{dx} = \frac{d(\tan^{-1}x)}{dx} - 0 = \frac{1}{1+x^2}$

4) If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ Prove that $\frac{dy}{dx} = \frac{4}{1+x^2}$

Sol^n: Let $x = \tan\theta$ $\quad \theta = \tan^{-1}x$

$$\frac{2x}{1+x^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$$

$$\frac{1+x^2}{1-x^2} = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1}{\cos 2\theta} = \sec 2\theta$$

$$\therefore y = \sin^{-1}(\sin 2\theta) + \sec^{-1}(\sec 2\theta)$$
$$= 2\theta + 2\theta$$
$$= 4\theta$$
$$y = 4\tan^{-1}x \qquad \frac{dy}{dx} = 4\frac{d}{dx}(\tan^{-1}x) = \frac{4}{1+x^2}.$$

5) If $y = \sin^{-1}2x\sqrt{1-x^2} + \sec^{-1}\frac{1}{\sqrt{1-x^2}}$, then prove that $\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

Sol^n: Let $x = \sin\theta$ $\quad \theta = \sin^{-1}x$

$$y = \sin^{-1}(2\sin\theta\cdot\cos\theta) + \sec^{-1}(\sec\theta)$$
$$= \sin^{-1}(\sin 2\theta) + \sec^{-1}(\sec\theta)$$
$$= 2\theta + \theta$$
$$= 3\theta$$
$$= 3\sin^{-1}x \qquad \frac{dy}{dx} = 3\times\frac{1}{\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}.$$

Miscallaneous Examples

1) Differentiate $\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$ w.r.t. x.

Soln: $y = \tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right) = \tan^{-1}\left(\frac{2\,(2\sqrt{x})}{1-(2\sqrt{x})^2}\right) = 2\tan^{-1}(2\sqrt{x})$

We know $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Let $2\sqrt{x} = \tan\theta$ then, $y = \tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]$

$y = \tan^{-1}(\tan 2\theta) = 2\theta$

$\therefore y = 2\tan^{-1}(2\sqrt{x})$

$$\frac{dy}{dx} = 2\,\frac{1}{1+(2\sqrt{x})^2}\,\frac{d(2\sqrt{x})}{dx} = \frac{2}{1+4x}\cdot\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x}}$$

$$= \frac{2}{\sqrt{x}\,(1+4x)}$$

[We can also use $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

2) Differentiate w.r.t x : (i) $\tan^{-1}\left(\frac{\sqrt{x}\,(3-x)}{1-3x}\right)$ (ii) $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

Soln: (i) $y = \tan^{-1}\left(\frac{\sqrt{x}\,(3-x)}{1-3x}\right) = \tan^{-1}\left[\frac{3\sqrt{x}-(\sqrt{x})^3}{1-3(\sqrt{x})^2}\right] = 3\tan^{-1}\sqrt{x}$

$\left[\because \tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}\right] \Rightarrow 3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$y = 3\tan^{-1}\sqrt{x}$

$$\frac{dy}{dx} = 3\,\frac{1}{1+x}\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2(1+x)\sqrt{x}}$$

(ii) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ put $x = \sin\theta$ $y = \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right)$

$$y = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\frac{\cancel{2}\sin\theta/2\cdot\cos\frac{\theta}{2}}{\cancel{2}\cos^2\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{\sin^{-1}x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}\,\frac{1}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}$$

3) Find the derivatives

(i) $\tan^{-1}\left(\dfrac{x^{\frac{1}{3}}+a^{\frac{1}{3}}}{1-x^{\frac{1}{3}}\cdot a^{\frac{1}{3}}}\right)$ (ii) $\sin^{-1}\left(\dfrac{3x+4\sqrt{1-x^2}}{5}\right)$

(i) $y=\tan^{-1}\left(\dfrac{x^{\frac{1}{3}}+a^{\frac{1}{3}}}{1-x^{1/3}a^{1/3}}\right)$ put $\tan\alpha = x^{\frac{1}{3}}$ $\tan\beta = a^{\frac{1}{3}}$

$\therefore y = \tan^{-1}\left(\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}\right) = \tan^{-1}\{\tan(\alpha+\beta)\}$

$y = \alpha+\beta =$

$= \tan^{-1}x^{\frac{1}{3}} + \tan^{-1}a^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{1+(x^{\frac{1}{3}})^2}\cdot\dfrac{d}{dx}(x^{1/3}) + 0 = \dfrac{1}{1+x^{2/3}}\cdot\dfrac{1}{3}\cdot x^{-2/3} = \dfrac{1}{3x^{2/3}(1+x^{2/3})}$

(ii) $y = \sin^{-1}\left(\dfrac{3x+4\sqrt{1-x^2}}{5}\right)$ put $x=\sin\theta$ $\theta = \sin^{-1}x$

$\therefore y = \sin^{-1}\left[\dfrac{3\sin\theta+4\cos\theta}{5}\right] = \sin^{-1}\left[\frac{3}{5}\sin\theta+\frac{4}{5}\cos\theta\right]$

Let $\frac{3}{5}=\cos\alpha$ $\frac{4}{5}=\sin\alpha$ then $\tan\alpha = \frac{4}{3}$ $\alpha = \tan^{-1}\frac{4}{3}$

$y = \sin^{-1}\{\cos\alpha\sin\theta+\sin\alpha\cos\theta\} = \sin^{-1}\{\sin(\theta+\alpha)\} = \theta+\alpha$

$y = \sin^{-1}x + \tan^{-1}\frac{4}{3}$ $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

4) Find derivative : $\tan^{-1}\left(\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}\right)$

[Hint.. $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}$ Ans. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)}$

5) If $y = \tan^{-1}\left(\dfrac{5ax}{a^2 - 6x^2}\right)$ Prove that $\dfrac{dy}{dx} = \dfrac{3a}{a^2+9x^2} + \dfrac{2a}{a^2+4x^2}$

[Hint... $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$]

6) If $y = \sin^{-1}[\,x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\,]$. Find $\dfrac{dy}{dx}$

[Hint.. $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$

Ans. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x-x^2}}$

7) Express the equation $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ in terms of an equation of inverse trigonometric function by suitable substitution and prove that

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$ [Hint. put $x = \sin\theta$, $y = \sin\phi$]

8) If $y = \sin^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) + \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, then show that

$\dfrac{dy}{dx} = \dfrac{-1}{2(1+x^2)}$ [Hint.. put $x = \tan\theta$]

9) If $y = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{1}{2}$.

10) If $y = b\tan^{-1}\left(\dfrac{x}{a} + \tan^{-1}\dfrac{y}{x}\right)$. find $\dfrac{dy}{dx}$ $\left[\dfrac{\dfrac{1}{a} - \dfrac{y}{x^2+y^2}}{\dfrac{1}{b}\sec^2\dfrac{y}{b} - \dfrac{x}{x^2+y^2}}\right]$

11) If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3-y^3)$, prove that

$\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$ [Hint.. put $x^3 = \sin\theta$, $y^3 = \sin\phi$

Derivatives of Implicit Functions.

If in a function the dependent variable y can be be explicitly written in terms of independent variable x i.e. term of 'x' must not involve in any manner then the function is called an explicit function e.g.

$y = x^2+1$, $y = \sin x + \cos x$

If the dependent variable y and independent variable x are so convoluted in an equation that y cannot be written explicitly as function of x then $f(x)$ is said to be an implicit function.

e.g. $x^2+y^2 = \tan^{-1} xy$. & $x^3+y^3 = 3axy$.

Here it is very difficult to express y as a function of x explicitly.

Implicit Differentiation

Find $\frac{dy}{dx}$ in the following :

(i) $2x + 3y = \sin x$

Differentiating both sides, we get

$$2(1) + 3\frac{dy}{dx} = \cos x$$

$$3\frac{dy}{dx} = \cos x - 2$$

$$\frac{dy}{dx} = \frac{1}{3}(\cos x - 2)$$

(ii) $2x + 3y = \sin y$

Differentiating both sides, we get

$$2(1) + 3\frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$(\cos y - 3)\frac{dy}{dx} = 2 \quad ; \quad \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

(iii) $x^2 + xy + y^2 = 100$

Differentiating both sides, we get

$$2x + x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} + 2x + y = 0$$

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

(iv) $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides, we get

$$3x^2 + x^2\frac{dy}{dx} + y\,2x + x\,2y\frac{dy}{dx} + y^2(1) + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 + 2xy + y^2 + (x^2 + 2xy + 3y^2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}$$

(v) $ax + by^2 = \cos y$

$$a(1) + b\,2y\frac{dy}{dx} = -\sin y\frac{dy}{dx} \quad ; \quad (2by + \sin y)\frac{dy}{dx} = -a$$

$$\frac{dy}{dx} = -\frac{a}{2by + \sin y}$$

Examples

1) Find $\frac{dy}{dx}$, when $xy^3 - yx^3 = x$ [ANS. $\frac{dy}{dx} = \frac{1-y^3+3x^2y}{3xy^2-x^3}$]

2) If $(x^2+y^2)^2 = xy$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$]

3) If $x^4+y^4-a^2xy=0$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{a^2y-4x^3}{4y^3-a^2x}$]

4) If $xy = 1+\log y$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{-y^2}{\log y}$].

5) If $x^3+y^3 = 3axy$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$]

6) If $2^x+2^y = 2^{x+y}$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{2^x(2^y-1)}{2^y(1-2^x)}$]

7) If $\sin(xy) = x\cos y$, find $\frac{dy}{dx}$ [ANS. $\frac{dy}{dx} = \frac{\cos y - y\cos xy}{x\cos xy + x\sin y}$]

8) If $y = x\sin y$, prove that $x\frac{dy}{dx} = \frac{y}{1-x\cos y}$ or $\frac{dy}{dx} = \frac{y}{x(1-x\cos y)}$

9) If $ax^2+2hxy+by^2+2gx+2fy+c=0$, find $\frac{dy}{dx}$

[ANS. $\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$]

10) If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \infty}}}$ Prove that $(2y-1)\frac{dy}{dx} = \frac{1}{x}$

[HINT.. $y = \sqrt{\log x + y}$]

11) If $\sin y = x\sin(a+y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Derivatives of functions in Parametric forms

The relationship between the variables x and y can be defined in parametric form using two equation:

$$x = f(t) \quad \text{and} \quad y = g(t)$$

Examples: $x = 2t+1$, $y = 4t-3$
$x = a\cos t$, $y = b\sin t$

When x and y are functions of the parametric t,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Solved Examples

1) Find $\frac{dy}{dx}$, when $x = a\sec\theta$ $y = b\tan\theta$

2) Find $\frac{dy}{dx}$, when $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at $\theta = \frac{\pi}{4}$

Ans. (i) $\frac{dy}{dx} = \frac{b}{a\sin\theta}$ (ii) $\frac{dy}{dx} = \tan\theta$ $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = 1$

3) Find $\frac{dy}{dx}$, where x and y are given by

$$x = 3\cos t - 2\cos^3 t \quad \text{and} \quad y = 3\sin t - 2\sin^3 t$$

4) Find $\frac{dy}{dx}$, when $x = a\sin 2\theta(1+\cos 2\theta)$
$y = b\cos 2\theta(1-\cos 2\theta)$

Ans. (3) $\cot t$ (4) $\frac{b}{a}\tan\theta$

Logarithmic Differentiation

The process of taking logarithms before differentiation is called logarithmic Differentiation. When the function consists of a single term which is a variable raised to a variable power, we first take logarithms and then differentiate.

Let us differentiate u^v when u and v are both functions of x.

Let $y = u^v$

Taking logarithms of both sides,

$$\log y = v \log u.$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = v\frac{1}{u}\frac{du}{dx} + \log u \frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$$

$$\frac{dy}{dx} = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \cdot \frac{dv}{dx}\right]$$

Remember:

1) When the function, which is to be differentiated, is of the form $f(x) = [u(x)]^{v(x)}$ where both u and v are functions of x, neither the formula for x^n nor that for a^x is applicable
2) In such cases, it is necessary to take its logarithm and then differentiate. Such a process is called logarithmic differentiation $\log y = v(x) \log [u(x)]$
3) In the case of a function consisting of a number of factors e.g. (i) $(x+1)^2(x+2)^3(x+3)$

Logarithmic differentiation can also be applied to a function which is again the product or quotient of two or more functions.

Solved Examples

1) Differentiate : $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Sol^n: let $y = \left[\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]^{\frac{1}{2}}$

Taking logarithm on both sides, we get

$$\log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Differentiate both sides w.r.t. x :

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

2) Find $\frac{dy}{dx}$, if $y = \dfrac{x(1-x^2)^{1/2}}{(1-x)}$

Sol^n: Given $y = \dfrac{x(1-x^2)^{1/2}}{(1-x)}$

$$\therefore \log y = \log x + \frac{1}{2}\log(1-x^2) - \log(1-x)$$

differentiating both side.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\frac{1}{(1-x^2)}(-2x) - \frac{1(-1)}{1-x} = \frac{1}{x} - \frac{x}{1-x^2} + \frac{1}{1-x}$$

$$\frac{dy}{dx} = y\left[\frac{1-x^2-\cancel{x^2}+x+\cancel{x^2}}{x(1-x^2)}\right] = \frac{\cancel{x}\sqrt{1-x^2}}{1-x}\left[\frac{1+x-x^2}{\cancel{x}(1-x^2)}\right]$$

$$\frac{dy}{dx} = \frac{\cancel{\sqrt{1-x^2}}\,(1+x-x^2)}{(1-x)\sqrt{1-x^2}\cdot\cancel{\sqrt{1-x^2}}} = \frac{1+x-x^2}{(1-x)\sqrt{1+x}\cdot\sqrt{1-x}} = \frac{1+x-x^2}{(1-x)^{3/2}(1+x)^{1/2}}$$

3) Find the derivatives of $\frac{\sqrt{x}(x+4)^{\frac{3}{2}}}{(4x-3)^{4/3}}$

Sol^n: Let $y = \frac{\sqrt{x}(x+4)^{\frac{3}{2}}}{(4x-3)^{4/3}}$

$$\log y = \log (x)^{\frac{1}{2}} + \log (x+4)^{3/2} - \log (4x-3)^{4/3}$$

$$= \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

Differentiating both sides,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x+4} - \frac{4}{3}\cdot\frac{1}{4x-3}\cdot 4$$

$$\frac{dy}{dx} = y\left[\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right]$$

$$\frac{dy}{dx} = \frac{\sqrt{x}(x+4)^{\frac{3}{2}}}{(4x-3)^{4/3}}\left[\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right]$$

4) Differentiate the following function w.r.t. x :

(i) x^x (ii) $x^{\frac{1}{x}}$ (iii) $x^{\sqrt{x}}$ (iv) $\left(\frac{1}{x}\right)^x$ (v) $x^{(x^x)}$

Sol^n (i) $y = x^x$ $\quad \log y = x\log x$

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot(1) = 1+\log x$$

$$\frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$$

(ii) $y = x^{\frac{1}{x}}$ $\qquad \log y = \frac{1}{x}\log x = \frac{\log x}{x}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x.\frac{d(\log x)}{dx} - \log x\frac{d(x)}{dx}}{x^2} = \frac{x\cdot\frac{1}{x} - \log x}{x^2}$$

$$\frac{dy}{dx} = y\frac{(1-\log x)}{x^2} = \frac{x^{\frac{1}{x}}(1-\log x)}{x^2}$$

(iii) $y = x^{\sqrt{x}}$ $\qquad \log y = \sqrt{x}\log x$

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\cdot\frac{1}{x} + \log x\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}}$$

$$\frac{dy}{dx} = x^{\sqrt{x}}\frac{[2+\log x]}{2\sqrt{x}}$$

(iv) $y = \left(\frac{1}{x}\right)^x$ $\qquad \log y = x\log\left(\frac{1}{x}\right) = x\log x^{-1}$

$$y = -x\log x$$

$$\frac{1}{y}\frac{dy}{dx} = -x\frac{1}{x} + \log x(-1) = -(1+\log x)$$

$$\frac{dy}{dx} = -\left(\frac{1}{x}\right)^x(1+\log x)$$

(v) $y = x^{x^{(x)}}$ $\qquad \log y = x^x\log x$

$$\log(\log y) = \log x^x + \log(\log x)$$

$$\log(\log y) = x\log x + \log(\log x)$$

Differentiating : $\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x + \frac{1}{\log x}\cdot\frac{1}{x}$

$$= 1+\log x + \frac{1}{x\log x}$$

$$\frac{dy}{dx} = y \log y \left[1 + \log x + \frac{1}{x\log x}\right]$$

$$= x^{(x^x)} \cdot x^x \log x \left[1 + \log x + \frac{1}{x\log x}\right]$$

$$\frac{dy}{dx} = x^{(x^x)} \cdot x^x \left[\log x + (\log x)^2 + \frac{1}{x}\right].$$

5) Differentiate : (i) $\cos x \cdot \cos 2x \cdot \cos 3x$

(ii) $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

(iii) $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Soln: (i) $y = \cos x \cdot \cos 2x \cdot \cos 3x$

$$\log y = \log \cos x + \log \cos 2x + \log \cos 3x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-\sin 2x)(2) + \frac{1}{\cos 3x}(-\sin 3x)(3)$$

$$= -\tan x - 2\tan 2x - 3\tan 3x$$

$$\frac{dy}{dx} = -y\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

$$\frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x \left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

(ii) $y = (x+3)^2 \cdot (x+4)^3 (x+5)^4$

$$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = 2 \times \frac{1}{x+3} + 3 \times \frac{1}{x+4} + 4 \times \frac{1}{x+5}$$

$$\frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

(iii) $y = x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

put $u = x^{x\cos x}$ $\quad v = \dfrac{x^2+1}{x^2-1}$

$\log u = x\cos x \log x$

Differentiating w.r.t. x,

$$\frac{1}{u}\frac{du}{dx} = \cos x \cdot \log x\,(1) + x\cos x\,\frac{1}{x} + x\log x\,(-\sin x)$$

$$= \cos x \log x - x\sin x \log x + \cos x$$

$$\frac{du}{dx} = x^{x\cos x}\left[\cos x \log x - x\sin x \log x + \cos x\right]$$

$$v = \frac{x^2+1}{x^2-1} \Rightarrow \frac{dv}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dv}{dx} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{x\cos x}\left[\cos x \cdot (1+\log x) - x\sin x \log x\right] - \frac{4x}{(x^2-1)^2}$$

6) Differentiate : (i) $(\sin x)^{\cos x}$ $\quad$ (ii) $(\sin x)^{\log x}$

Ans (i) $\dfrac{dy}{dx} = (\sin x)^{\cos x}\left[\cot x \cdot \cos x - \sin x \log \sin x\right]$

(ii) $\dfrac{dy}{dx} = (\sin x)^{\log x}\left[\log x \cot x + \dfrac{1}{x}\log \sin x\right]$

Miscellaneous Examples

1) Differentiate the following :

(i) $(x^2 \sin x)^{-\frac{1}{x}}, x \neq 0$ Ans : $(x^2 \sin x)^{-\frac{1}{x}}\left[\frac{1}{x^2}\left\{(2 + x\cot x - \log(x^2 \sin x)\right\}\right]$

(ii) $(\log x)^{\tan x}$ Ans: $(\log x)^{\tan x}\left[\frac{\tan x}{x\log x} + \sec^2 x \log(\log x)\right]$

(iii) $(1+x)^{\log x}, x>0$ Ans: $(1+x)^{\log x}\left[\frac{\log(1+x)}{x} + \frac{\log x}{1+x}\right]$

(iv) $(\tan x)^{\log x}$ Ans: $(\tan x)^{\log x}\left[\frac{\log x}{\sin x \cdot \cos x} + \frac{\log \tan x}{x}\right]$

2) Differentiate the following w.r.t. x :

(i) $(\sin^{-1}x)^x$ Ans: $(\sin^{-1}x)^x\left[\frac{x}{\sqrt{1-x^2}\sin^{-1}x} + \log(\sin^{-1}x)\right]$

(ii) $x^{\sin^{-1}x}$ Ans: $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$

(iii) $(\sin x)^{\cos^{-1}x}$ Ans: $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x \cdot \cot x - \frac{\log(\sin x)}{\sqrt{1-x^2}}\right]$

(iv) $x^{\cos^{-1}x}$ Ans: $x^{\cos^{-1}x}\left[\frac{1}{x}\cos^{-1}x - \frac{\log x}{\sqrt{1-x^2}}\right]$

3) (i) Differentiate : $x^x \sin^{-1}\sqrt{x}$ Ans: $x^x \sin^{-1}\sqrt{x}\,(1+\log x) + \frac{x^x}{2\sqrt{x-x^2}}$

(ii) Differentiate : $\cos(x^x)$ Ans: $-\sin(x^x) \cdot x^x(1+\log x)$.

4) (i) Differentiate $x^{\sin 2x + \cos 2x}$

Ans : $x^{\sin 2x + \cos 2x}\left[\frac{\sin 2x + \cos 2x}{x} + 2(\cos 2x - \sin 2x)\cdot \log x\right]$

(ii) Differentiate x^{x^2} Ans: $x^{x^2}(x + 2x\log x) = x^{x^2+1}(1+2\log x)$

5) (i) If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, find $\frac{dy}{dx}$

Ans: $x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\cos x)^{\sin x}(-\tan x \cdot \sin x + \cos x \log \cos x)$

(ii) If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, find $\frac{dy}{dx}$

Ans: $(\sin x)^{\tan x}[1 + \log(\sin x) \cdot \sec^2 x] + (\cos x)^{\sec x} \cdot \sec x \tan x [\log(\cos x) - 1]$

6) Differentiate $(\sin x)^x + \sin^{-1}\sqrt{x}$ w.r.t. x

Ans: $(\sin x)^x [x \cot x + \log(\sin x)] + \frac{1}{2\sqrt{x - x^2}}$

7) If $y = (\cos x)^{\log x} + (\log x)^x$, find $\frac{dy}{dx}$

Ans: $(\cos x)^{\log x}\left[\frac{\log(\cos x)}{x} - \log x \cdot \tan x\right] + (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$

8) Differentiate the following: $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Ans: $(x \cos x)^x [1 - x \tan x + \log(x \cos x)] + (x \sin x)^{\frac{1}{x}}\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$

9) Differentiate the following: $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$

Ans. $\left(x + \frac{1}{x}\right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right] + x^{\left(1 + \frac{1}{x}\right)}\left[\frac{x + 1}{x^2} - \frac{\log x}{x^2}\right]$

10) If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$ Ans: $\frac{\log(\sin y) + y \tan x}{\log \cos x - x \cot y}$

11) If $y^x + x^y + x^x = a^b$, find $\frac{dy}{dx}$

Ans: $-\frac{\{y^x(\log y) + y \cdot x^{(y-1)} + x^x(1 + \log x)\}}{\{xy^{(x-1)} + x^y(\log x)\}}$

Second Order Derivative

The derivative $\frac{dy}{dx}$ is called the first differential co-efficient or first order derivative of y w.r.t. x.

The differential co-efficient of $\frac{dy}{dx}$ i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second differential co-efficient or second order derivatives of y.

Which we denote as $\frac{d^2y}{dx^2}$ [read dee two y over dee x squared]

Solved Examples:

1) If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

2) If $y = a\cos(\log x) + b\sin(\log x)$, prove that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

3) If $y = \tan x$, prove that $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$

4) If $y = \sin^{-1}x$, prove that $\frac{d^2y}{dx^2} = \frac{x}{(1-x^2)^{3/2}}$

5) If $y = A\sin mx + B\cos mx$, Show that $\frac{d^2y}{dx^2} + m^2y = 0$

6) If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Rolle's Theorem

Statement: If a function $f(x)$ is such that

(i) it is continuous in the closed interval $[a, b]$

(ii) it is differentiable in the open interval $]a, b[$

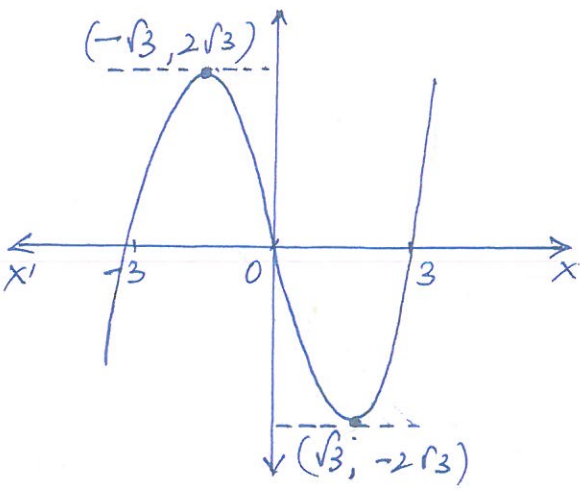
(iii) $f(a) = f(b)$

then there exists at least one value 'c' of x in the open interval (a, b) such that $f'(c) = 0$.

for example, $f(x) = \frac{x^3}{3} - 3x$

is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$.

We have $f(-3) = f(3) = 0$.



$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

Rolle's Theorem says that f' must be zero at least once in the open interval $(-3, 3)$.

$$f'(x) = \frac{dy}{dx} = x^2 - 3 \qquad f'(x) = 0$$

$$x^2 - 3 = 0 \qquad x = \pm\sqrt{3}$$

So $f'(x)$ is zero twice in the interval, at $x = -\sqrt{3}, \sqrt{3}$.

Thus, there exists $c = -\sqrt{3}, \sqrt{3} \in (-3, 3)$ such that $f'(c) = 0$.

Solved Examples

1) Verify Rolle's Theorem for the function $f(x) = x^2 - 5x + 4$ on $[1,4]$.

Sol^n : $f(x) = x^2 - 5x + 4$

$f(x)$ being a polynomial function is continuous in $[1,4]$.

$f'(x) = \frac{dy}{dx} = 2x - 5$ is differentiable in $(1,4)$

$f(1) = 1 - 5 + 4 = 0$

$f(4) = 16 - 5\times 4 + 4$

$= 16 - 20 + 4$ $\quad \therefore f(1) = f(4)$

$= 0$

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ There must exist at least one number c such that $1 < c < 4$ and $f'(c) = 0$.

Now, $f'(c) = 0 \quad 2c - 5 = 0 \quad c = \frac{5}{2} \in (1,4)$

Hence, Rolle's Theorem is verified.

2) Using Rolle's Theorem, find the point on the curve if $f(x) = x^2$, $x \in [-2,2]$ where tangent is parallel to the x-axis.

Sol^n : Since $f(x)$ is polynomial, therefore, it is continuous function.

$f'(x) = 2x$ which exists in $(-2,2)$

$\therefore f(x)$ is derivable in $(-2,2)$

Also, $f(-2) = f(2) = 4$

Thus $f(x)$ satisfies all conditions.

$f'(c) = 0$

$2c = 0$

$c = 0 \in (-2,2)$

3) verify Rolle's theorem for the function $f(x) = x(x-3)^2$
$0 \leq x \leq 3$

4) verify Rolle's Theorem for the function
$f(x) = \sqrt{1-x^2}$ in $[-1, 1]$

5) Discuss the applicability of Rolle's Theorem for the function
$f(x) = (x-1)^{2/3}$ in the interval $[0,3]$.

Solⁿ: $f(x) = (x-1)^{2/3}$

$f(x)$ being polynomial function is continuous in $[0,3]$

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

which does not exist at $x=1 \in (0,3)$

So, $f(x)$ is not derivable in $(0,3)$

Hence, Rolle's Theorem is not applicable.

6) It is given that for the function $f(x) = x^3 - 6x^2 + ax + b$
on $[1,3]$. Rolle's Theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$.
Find the values of a and b.

Solⁿ: $f(x) = x^3 - 6x^2 + ax + b$

Since Rolle's Theorem holds for $f(x)$
$f(x)$ is continuous in $[1,3]$ and derivable in $(1,3)$
which is obvious because $f(x)$ is a polynomial.

Also $f(1) = f(3)$

$$f(1) = (1)^3 - 6(1)^2 + a(1) + b$$
$$= a + b - 5$$

$$f(3) = 27 - 54 + 3a + b$$
$$= 3a + b - 27$$

$f(1) = f(3) \Rightarrow$ $a + \cancel{b} - 5 = 3a + \cancel{b} - 27$
$$2a = 22$$
$$a = 11$$

$$f'(x) = 3x^2 - 12x + 11$$

Since $f'(c) = 0$ $\quad c = 2 + \frac{1}{\sqrt{3}}$

$\therefore$ $3c^2 - 12c + 11 = 0$ which is independent of b

Hence $a = 11$ and b is arbitrary.

Miscellaneous Examples

1) Verify Rolle's Theorem :

(i) $f(x) = (x-a)^3 (x-b)^4$ in the interval $[a, b]$

(ii) $f(x) = (x-a)^m (x-b)^n$, $x \in [a, b]$ where m and m are positive integers.

2) Verify Rolle's Theorem for following function.

$$f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right] \text{ in } (a, b)$$

Lagrange's Mean Value Theorem

Rolle's theorem can be used to prove another theorem – the Mean Value Theorem.

Statement:

If a function $f(x)$ is such that
(i) it is continuous in the closed interval $[a, b]$

(ii) it is derivable in the open interval $]a, b[$ then there exists at least one value 'c' of x in the open interval $]a, b[$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

For example,
let $f(x) = x^2$, $a=2$ and $b=5$

Now, $f(x) = x^2$ is continuous for $2 \leq x \leq 5$ and differentiable for $2 < x < 5$

The mean value theorem says that at some point c is $(2, 5)$ the derivative $f'(x) = 2x$ must have the value

$$\frac{f(b)-f(a)}{b-a} = \frac{25-4}{5-2} = \frac{21}{3} = 7 = f'(c)$$

$$f'(c) = 2c$$

$$2c = 7 \Rightarrow c = \frac{7}{2}$$

Solved Examples

1) Verify the condition of Mean Value Theorem and in each case find a point c in the interval.

(i) $f(x) = 3x^2 - 2$ on $[2, 3]$ $\left(c = 5/2\right)$

(ii) $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$ $\left(c = 1/3\right)$

2) Verify Langrange's Mean Value Theorem for the following functions:

(i) $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$

(ii) $f(x) = \log x$ in $[1, e]$

(iii) $f(x) = e^x$ in $[0, 1]$

3) Verify Mean Value Theorem for the following function and find c, if possible.

$f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

[Ans $c = 1 - \frac{\sqrt{21}}{6}$]