Chapter 1 : Relations and Functions (XII)

Types of Relation

1) **Empty Relation** : A relation R in a set A is called empty relation, if no element of set A is related to any element of set A i.e. $R = \phi$. $R = \phi \subset A \times A$

Example, consider a relation R in the set $A = \{1, 2, 3, 4\}$

$R = \{(a, b) \mid a \in A, b \in A, a \neq b \text{ and } a = b\}$

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2) \ldots\}$

This is the empty set as no pair (a, b) satisfy the condition $a \neq b$ and $a = b$ simultaneously.

Again consider the relation R in the set $A = \{1, 3, 5, 7\}$ is given by $R = \{(a, b) : a - b = 9\}$

This is the empty set, as no pair of set A satisfies the condition $a - b = 9$ in A.

Let A = Set of all students in a girls school.

$R = \{(a, b) : a$ and b are brothers$\}$

It is a girl school, so there are no boys in the school. Hence, there cannot be a brother. $\therefore R$ has no elements.

$\Rightarrow R = \phi$ i.e. R is an empty relation.

2) **Universal Relation** :

A relation R in a set A is called universal relation, if each element of set A is related to every element of set A i.e. $R = A \times A$.

Example, let A = set of all student in a girls school

$R' = \{(a,b) :$ height of a & b is greater than 10 cm.

Height of children are always greater then 10cm (0.3 foot).
Then, height of a and b will always be greater then 10cm.
$\therefore R$ has all the students of the school.

$\Rightarrow R$ is a universal relation, since it has all the elements.

<u>Note</u>: Sometimes, empty relation and universal relation are called <u>trival relations</u>.

Consider relation R in the set $A = \{1, 3, 5, 7\}$ given by
$R = \{(a,b) : |a-b| \geq 0\}$

All pairs (a,b) in set $A \times A$ satisfies $|a-b| \geq 0$.
$\therefore R$ is a universal relation.

3) Identity Relation: The identity relation is a set of all ordered pairs $(a,b) \in A \times A$ such that $a = b$ denoted by I_A.
$\therefore I_A = \{(a,b) \mid a \in A, b \in A$ and $a = b\}$

Domain of I_A = Range of $I_A = A$.

Example, If $A = \{1, 2, 3, 4, 5, 6\}$ then $I_A = \{(1,1), (2,2), (3,3), (4,4)\}$

Note: In an identity relation on A, every element of A should be related to itself only.

4) **Inverse Relation**: Let R be a relation from A to B. In inverse relation of R, denoted by R^{-1} is a relation from set B to set A defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

R^{-1} consists of those ordered pairs which, when reversed belong to R. Example, $A = \{1,2,3,4\}$ $B = \{a,b,c\}$

$R = \{(1,a), (2,b), (3,a, (4,b)\}$ then
$R^{-1} = \{(a,1)\ (b,2), (a,3), (b,4)\}$

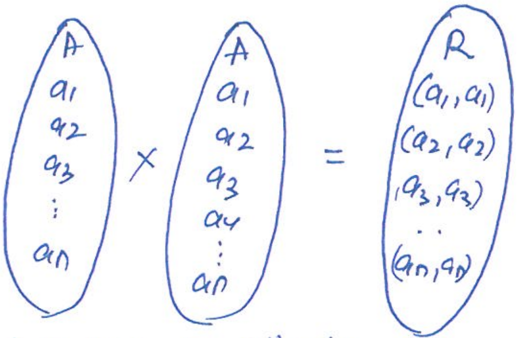
Domain of R^{-1} = Range of R
Range of R^{-1} = Domain of R.

Reflexive Relation

A relation R in a set A is said to be reflexive, if aRa for all $a \in A$ i.e. if $(a,a) \in R\ \forall\ a \in A$.
Let us take a relation R in a set A.

It is proven to be reflexive, if $(a,a) \in R$, for every $a \in A$.

Lets take any set $K = (2, 8, \}$
if Relation $R = \{(2,2), (8,8,), (9,9) \ldots\}$ exists, then relation R is called a Reflexive relation.



$R \subset A \times A$ is reflexive if $(a,a) \in R\ \forall\ a \in A$.

Let A be the set of triangles in a plane. The relation R in A "is similar to" is reflexive because every traingle is similar to itself.

A relation R in a set A is not reflexive if there is at least one element $a \in A$ such that $(a,a) \notin R$.

Symmetric Relation

A relation R in a set A is said to be symmetric if $aRb \Rightarrow bRa$ i.e. $(a,b) \in R \Rightarrow (b,a) \in R$.

Thus a relation R in set A is symmetric if and only if $R = R^{-1}$.

Consider, for example, the set A of natural numbers. If a relation A be defined by "$x+y=5$", then this relation is symmetric in A, for $a+b=5 \Rightarrow b+a=5$.

But in the set A of natural numbers if the relation R be defined as 'x is a divisor of y', then the relation R is not symmetric as 3R9 does not imply 9R3, for, 3 divides 9 but 9 does not divide 3.

Let A be the set of lines in a plane and let R mean "is perpendicular to" then R is symmetric, i.e., if a line a is perpendicular to the line b, then it implies that the line b is perpendicular to line a.
$aRb \Rightarrow bRa$.

Transitive Relation

A relation R is said to be transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$, That is aRb and $bRc \Rightarrow aRc$ where $a,b,c \in A$.

For example, in the set A of natural numbers if the relation R be defined by 'x less than y' then
$a<b$ and $b<c$ imply $a<c$, that is aRb and $bRc \Rightarrow aRc$

Let A be the set of all lines in a plane and R be the relation in set A defined by "is parallel to". Then if a line is parallel to line b and the line b is parallel to line c, then line a is parallel to line c.
Here R is transitive.

Equivalence Relation

A relation R on a set A is called an equivalence relation on A when R is reflexive, symmetric and transitive. Thus R is an equivalence relation if

(i) aRa, $\forall a \in A$ (R is reflexive)

(ii) $aRb \Rightarrow bRa$ $\forall a, b \in A$ (R is symmetric)

(iii) aRb and $bRc \Rightarrow aRc$ $\forall a, b, c \in A$ (R is transitive).

The symbol "$\sim$" (called wave or wiggle) is used for an equivalence relation.

Let A be the set of all triangles in a plane and let R be defined by "is congruent to".

(i) R is reflexive i.e. aRa, for every $a \in A$. Since every triangle is congruent to itself.

(ii) R is symmetric i.e. $aRb \Rightarrow bRa$. Since if triangle is congruent to triangle b then triangle b is congruent to triangle a.

(iii) R is transitive i.e. aRb and $bRa \Rightarrow aRc$. Since if triangle a is congruent to triangle b and triangle b is congruent to triangle c, then triangle a is congruent to triangle c.

Thus the relation as defined above is reflexive, symmetric and transitive. Hence, R is an equivalence equation.

Properties: (i) The inverse of an equivalence relation is also an equivalence relation. So, if R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(ii) The intersection of two equivalence relation on a set A is also an equivalence relation.

(iii) The union of two equivalence relation on a set A is not necessarily an equivalence relation on A.

Congruence Modulo m

Let m be a given positive integer i.e. $m \in N$ and $a, b \in Z$. Then the 'congruence modulo m' on set Z is defined by $a \cong b \pmod{m} \Rightarrow (a-b)$ is divisible by m.

$a \cong b \pmod{m}$ say that a is congruent to B modulo m.

Example, (i) $15 \cong 5 \pmod{5}$, since $(15-5) = 10$ is divisible by 5.

(ii) $24 \cong 30 \pmod{6}$, since $(24-30) = -6$ is divisible by 6.

(iii) $138 \cong -2 \pmod{5}$, since $(138-(-2)) = 140$ is divisible by 5.

(iv) $28 \ncong 34 \pmod{5}$, since $28-34 = -6$ is not divisible by 5.

Theorem: The relation 'congruence modulo m' is an equivalence relation in set Z of all integers.

Equivalence Classes

Let R be an equivalence relation on a set A, and let $a \in A$. The equivalence class of a is called the set of all elements of A which are equal to a.

The equivalence class of an element a is denoted by [a]. Thus $[a] = \{b \in A \mid aRb\} = \{b \in A \mid a \sim b\}$.

If $b \in [a]$ then the element b is called a representive of the equivalence class [a]. Any element of an equivalence class may be chosen as a representive of the class.

The set of all equivalence classes of A is called the quotient set of A by the relation R. The quotient set is denoted a A/R. $A/R = \{[a] \mid a \in A\}$.

Properties of Equivalence classes:

If R (also denoted by $\sim$) is an equivalence relation on set A, then

(i) Every element $a \in A$ is a member of the equivalence class $[a]$ $\forall a \in A$, $a \in [a]$.

(ii) Two elements $a, b \in A$ are equivalent if and only if they belong to the same equivalence class.
$\forall a, b \in A$, $a \sim b$ iff $[a] = [b]$.

(iii) Every two equivalences classes $[a]$ and $[b]$ are either equal or disjoint. $\forall a, b \in A$, $[a] = [b]$ or $[a] \cap [b] = \phi$.

Example

A well known sample equivalence relation is Congruence Modulo m. Two integers a and b are equivalent if they have same remainder after dividing by m.

Consider, for example, the relation of congruence modulo 3 on the set of integers $\mathbb{Z}$:

$$R = \{(a,b) \mid a \cong b \pmod 3\}$$

The possible remainders for $n = 3$ are 0, 1, and 2. An equivalence class consists of those integers that have the same remainder. Hence, there are 3 equivalence classes in this example:

$$[0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9 \ldots\}$$

$$[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10 \ldots\}$$

$$[2] = \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}$$

Note: Two equivalences classes are either identical or disjoint. Let R be an equivalence relation on a non-empty set A. $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \phi$.

Partitions of a Set

Let A be a set and $A_1, A_2, \ldots, A_n$ be its non-empty subsets. The subsets form a partition P of A if

i) The union of the subsets in P is equal to A.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = A$$

(ii) The partition P does not contain the empty set ϕ
$A_i \neq \phi \ \forall i$

(iii) The intersection of any distinct subsets in P is empty
$A_i \cap A_j = \phi \quad \forall i \neq j$

There is a direct link between equivalence classes and partitions. For any equivalence relation on a set A, the set of all its equivalence classes is a partition of A.

The converse is also true.

The following relations are equivalence relations :

i) "a and b live in the same city" on the set of all people
ii) "a and b are the same age" on the set of all people
(iii) "a and b have same remainder when divided by 3" on set of integers.
(iv) "a and b have the same last digit" on the set of integers.

Any relation that can be defined using expressions like "have the same" or "are the same" is an equivalence relation.

(v) "a and b are parallel lines" on the set of all straight lines of a plane.
(vi) "a and b are similar triangles" on the set of all triangles.

Solved Examples

(1) Determine whether each of the following relations are reflexive, symmetric or transitive.

(i) Relation R in the set $A = \{1, 2, 3, 4, \ldots 13, 14\}$ defined as
$$R = \{(x,y) : 3x - y = 0\}$$

(ii) Relation R in the set N of all natural numbers defined as $R = \{(x,y) : y = x+5 \text{ and } x < 4\}$

(iii) Relation R in the set Z of all integers defined as
$R = \{(x,y) : x-y \text{ is an integer}\}$.

Soln: (i) $R = \{(x,y) : 3x - y = 0\}$ $\quad 3x - y = 0 \; ; \; y = 3x$

$\therefore \; R = \{(1,3), (2,6), (3,9), (4,12)\}$

We see that for any $a \in A$, $(a,a) \notin R$.

But a relation R in a set is called reflexive if $(a,a) \in R$ for every $a \in A$. Hence, the relation R is not reflexive.

R is not symmetric because $(1,3) \in R$ but $(3,1) \notin R$.

R is not transitive because $(1,3) \in R$ and $(3,9) \in R$ but $(1,9) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

(ii) $R = \{(x,y) : y = x+5 \text{ and } x < 4\}$
$R = \{(1,6), (2,7), (3,8)\}$

R is not reflexive on set N because $1 \in N$ but $(1,1) \notin R$.
R is not symmetric because $(1,6) \in R$ but $(6,1) \notin R$.
R is not transitive because $(1,6) \in R$, $(6,11) \notin R$, $(1,11) \notin R$.

(iii) $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$R = \{(x, y) : x - y$ is an integer$\}$

We have $\forall x \in Z$, $x - x = 0 \in Z$ (0 is an integer)

$\Rightarrow (x, x) \in R \ \forall x \in Z$.

So, R is reflexive.

$(x, y) \in R \Rightarrow x - y$ is an integer

$\Rightarrow -(x - y)$ is an integer

$\Rightarrow y - x$ is an integer

$\Rightarrow (y, x) \in R$

$\therefore$ R is symmetric

Let $(x, y) \in Z$ such that $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow (x - y)$ is an integer and $(y - z)$ is an integer

$\Rightarrow (x - y) + (y - z)$ is an integer

$\Rightarrow (x - z)$ is an integer

$\Rightarrow (x, z) \in R$

$\therefore$ R is transitive.

Hence, R is reflexive, symmetric and transitive.

(2) Check whether the relation R in real numbers R defined by $R = \{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

Soln: Let $a = -\frac{1}{2}$ $b = -\frac{1}{2}$ $\Rightarrow b^3 = -\frac{1}{8}$ Since $-\frac{1}{2} \leq -\frac{1}{8}$ false

$\Rightarrow \left(-\frac{1}{2}, -\frac{1}{2}\right) \notin R$. $\therefore$ R is not reflexive.

If $a \leq b^3$, then b is not $\leq a^3$ $\therefore$ R is not symmetric.

If $a \leq b^3$ and $b \leq c^3$, then a is not necessarily $\leq c^3$.

R is not transitive.

Hence, relation R is neither reflexive, nor symmetric nor transitive.

3) Which of these relations on the set $A = \{1, 2, 3, 4\}$ are equivalence relations?

i) $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

(ii) $R_2 = \{(1,4), (2,2), (3,3), (4,1), (4,2), (4,4)\}$

(iii) $R_3 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\}$

(iv) $R_4 = \{(1,1), (1,2), (1,4), (2,2), (2,4), (3,3), (4,1), (4,4)\}$

Soln: (i) $1 \in A$ and $(1,1) \in R$, R is reflexive

$(1,2) \in R$ and $(2,1) \in R$, R is symmetric.

$(1,1) \in R$, $(1,2) \in R$ and $(1,2) \in R$, R is transitive.

R_1 is equivalence relation.

(ii) $1 \in A$ but $(1,1) \notin R_2$, $\therefore R_2$ is not reflexive

$(4,2) \notin R_2$ but $(2,4) \notin R_2$, R_2 is not symmetric

$(1,4), (4,2) \in R_2$ but $(1,2) \notin R_2$, R_2 is not transitive.

R_2 is not an equivalence relation.

(iii) R_3 is an equivalence relation since it is reflexive, symmetric and transitive.

(iv) R_4 is not symmetric since $(1,2) \in R_4$, but $(2,1) \notin R_4$. Similarly $(2,4) \in R_4$ but $(4,2) \notin R_4$. Thus R_4 not an equivalence relation.

(4) Which of these relations on the set of all people are equivalence relation?

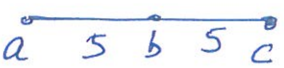
i) a & b speak the same language.
(ii) a and b speak a common language.
(iii) a and be have same mother
(iv) a lives with 5 miles of b
(v) a loves b.
(vi) a and b are younger than 20.
(vii) a is older than b.

Soln: (i) This is an equivalence relation.
a speak the same language, so this relation is reflexive.
If a speaks the same language as b, b speaks the same language as a, so this relation is symmetric.
If a speaks the same language as b and b speak the same language as c, then a speaks the same language c.
Thus this relation is transitive.

(ii) This relation is reflexive and symmetric, but not transitive.
Example, a and b speak a common language, say French, and b speak another common language, say German. This means that a and c may not have a common language.
∴ This relation is not an equivalence relation.

(iii) This is an equivalence relation.

(iv) This relation is reflexive and symmetric, but it is not transitive.



a to b = 5 m b to c = 5 mile
but a to c ≠ 5 mile.

(iv) a loves b.

This relation is not reflexive.
Though many people love themselves, this does not mean that this property is true for all people in the relation. Similarly, if a love b, then it may be that b love a, but it may also not be. So the relation is not symmetric. It is easy to see that the relation is not transitive.

(v) This is an equivalence relation

(vi) This relation is not reflexive : a as not older than itself. This is not symmetric : if a is older than b, converse is false. This relation is transitive : if a is older than b and b is older than c, then a is older than c.

∴ This is not an equivalence relation.

(5) Which of the following collections of subsets are partitions of $\{0,1,2,3,4,5\}$?

a) $\{0,1,2\}$, $\{4,3\}$, $\{5,4\}$
(b) $\{\}$, $\{0,2,1\}$, $\{4,3,5\}$
(c) $\{5,4,0,3\}$, $\{2\}$, $\{1\}$
(d) $\{5\}$, $\{4,3\}$ $\{0,2\}$
(e) $\{2\}$, $\{1\}$, $\{5\}$, $\{3\}$, $\{0\}$, $\{4\}$

Soln : a) $A_1 = \{0,1,2\}$ $A_2 = \{4,3\}$ $A_3 = \{5,4\}$

$A_1 \cup A_2 \cup A_3 = A$ $A_i \neq \phi$

But $A_2 \cap A_3 = 4$ $[A_2 \cap A_3 = \phi$ for partition set$]$

∴ This is not partition.

b) Not a partition because they have the empty set.

c) This is a partition

d) Not a partition because their union is not A.

e) form a partition.

6) List all the partition of the following sets:
a) $A = \{1, 2\}$ (b) $B = \{1, 2, 3\}$

Solⁿ: (i) $A = \{1, 2\}$ has 2 partition:
$\{1\}, \{2\}$ and $\{1, 2\}$

(ii) $B = \{1, 2, 3\}$ has 5 partitions:
$\{1\}, \{2\}, \{3\}$. $\{1\}, \{2, 3\}$

$\{1, 2\}, \{3\}$ $\{1, 2, 3\}$

$\{1, 3\}, \{2\}$

7) List the ordered pairs in the equivalence relation R induced by the partition $P = \{\{a, b, c\}, \{d\}, \{e\}\}$ of the set $\{a, b, c, d, e\}$.

Solⁿ: The partition P includes 3 subsets which corresponds to 3 equivalence classes of the relation R.

We can denote these classes by E_1, E_2 and E_3.
They contain the following pairs:
$E_1 = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, a), (c, b), (c, c)\}$
$E_2 = \{d, d\}$ $E_3 = \{e, e\}$

So relation R in roster form $R = E_1 \cap E_2 \cap E_3$
$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (e, e)\}$

8) A relation R on the set $A = \{a, b, c, d, e\}$ is defined as follows:

$$R = \{(a,a), (a,b), (b,a), (b,b), (c,c), (c,d), (c,e), (d,c)$$
$$(d,d), (d,e), (e,c), (e,d), (e,e)\}.$$

Determine the equivalence classes of R.

Soln: First we check that R is an equivalence relation.

R is reflexive since it contain all identity elements $(a,a), (b,b) \ldots$

R is symmetric. $(a,b)\ (b,a) \in R$ $(c,d)\ (d,c) \in R$.

R is transitive. Example, the relation contains the overlapping pairs $(a,b), (b,a)$ and the element (a,a).

Thus we conclude that R is an equivalence relation.

Consider the elements related to a. The relation R contains the pairs (a,a) and (a,b). Hence a and b are related to a.
Similarly, we find that a and b are related to b.
So these items form the equivalence class $\{a, b\}$.
Notice that the relation R has $2^2 = 4$ ordered pair within this class.

Take the next element c and find all elements related to it.
There are 3 pairs within the first element c: $(c,c), (c,d)\ (c,e)$.

Similarly, we find pairs with the elements related to d and e: $(d,c), (d,d), (d,e), (e,c), (e,d)$ and (e,e).
This set of $3^2 = 9$ pairs corresponds to the equivalence class $\{c, d, e\}$ of 3 elements.

Thus, the relation R has 2 equivalance classes $\{a, b\}$ and $\{c, d, e\}$.

9) The relation $R = \{(a,b) : |a+1| = |b+1|\}$ is defined on the set of integers Z. Find the equivalence classes for R.

Soln: It's easy to make sure that R is an equivalence relation.

The equivalence classes of R are defined by the expression $\{-1-n, -1+n\}$, where n is an integer.

$n = 0$: $E_0 = [-1] = \{-1\}$, $R_0 = \{(-1,-1)\}$

$n = 1$: $E_2 = [-2] = \{-2, 0\}$, $R_1 = \{(-2,-2), (-2,0), (0,-2), (0,0)\}$

$n = 3$: $E_3 = [-3] = [-3, 1]$, $R_2 = \{(-3,-3), (-3,1)\ (1,-3), (1,1)\}$

$n = -2$, $E_{-2} = [1] = [1,-3]$, $R_{-2} = \{(1,1), (1,-3), (-3,1), (-3,-3)\}$

As it can be seen that $E_2 = E_{-2}$, $E_{10} = E_{-10}$.

It follows from here that we can list all equivalence classes for R by using non-negative integers n.

Functions

A function is a special relation : a set of ordered pairs in which no two distinct ordered pairs have the same first element.

In the relation $y = x^2$ $(2,4), (-2,4), (3,9), (-3,9) \ldots$
For each value of x there is only one corresponding value of y.
Therefore, $y = x^2$ is a function.

In the relation $x = y^2$ $\underline{(4,2)}, \underline{(4,-2)}, (9,3), (9,-3) \ldots$
We see that two ordered pairs have same first element.
Therefore, $x = y^2$ is not a function.

For example, $A = \pi r^2$, the area of circle depends on its radius.
We say that "A is a function of r".

Definition :

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.
If f is a function from A to B and $(a,b) \in f$, then $f(a) = b$, where "b" is called the image of "a" under f and a is called the pre-image of "b" under f.

Let $f : A \to B$, then set A is called domain
set B is called co-domain.

Example :

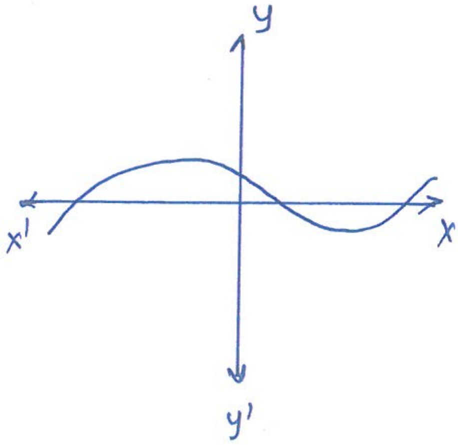
$A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 5, 6, 11, 12, 18\}$
A function $f : A \to B$ is defined by $f(x) = x^2 + 2$,
$f(1) = 3$, $f(2) = 6$ $f(3) = 11$ etc.

(i) $f = \{(1,3), (2,6), (3,11), (4,18)\}$
(ii) Domain of $f = A$, Co-Domain of $f = B$
(iii) Range of $f = \{3, 6, 11, 18\}$

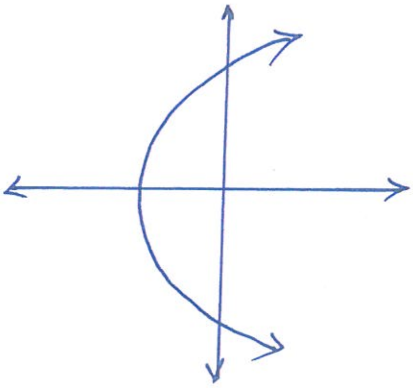
Graph of Functions

The graph of a function is the set of points $(x, f(x))$ such that x is in the domain of f. The y- co-ordinate of any point (x, y) on the graph is $y = f(x)$.

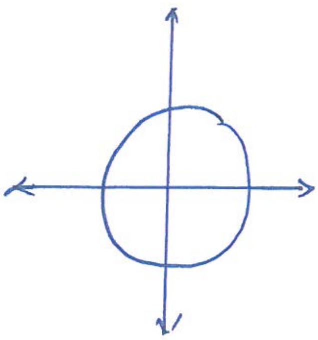
If there are two or more points of graph on same vertical line, then the relation is not a function.



(i)



(ii)



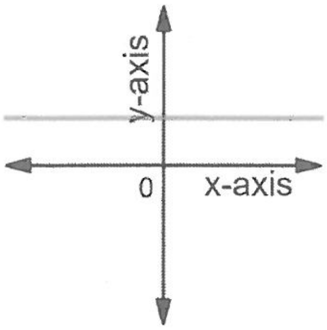
(iii)

In Fig (i) no vertical line intersects the graph more than once. So, it is a function.

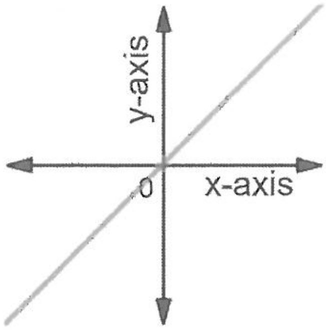
In Fig (i) & (ii) vertical line meets the graph in more than one point. So, these do not represent functions.

Class - 12

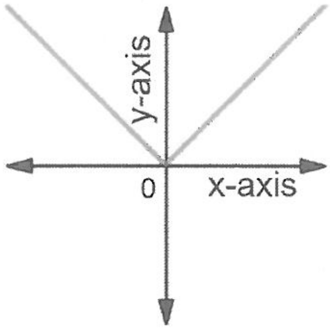
Constant Function: $f(x) = 2$



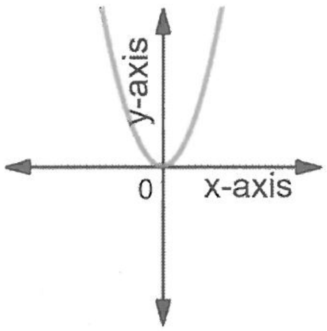
Identity: $f(x) = x$



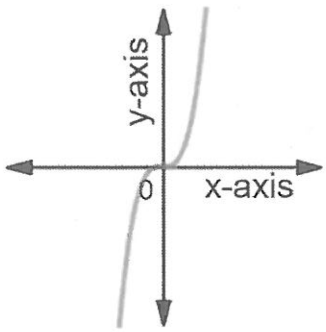
Absolute Value: $f(x) = |x|$



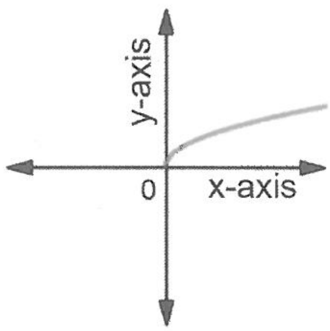
Quadratic: $f(x) = x^2$



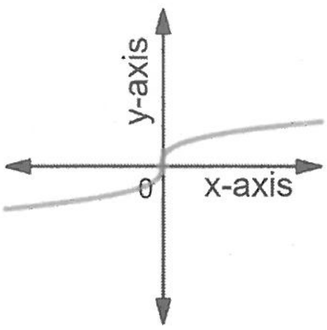
Cubic: $f(x) = x^3$



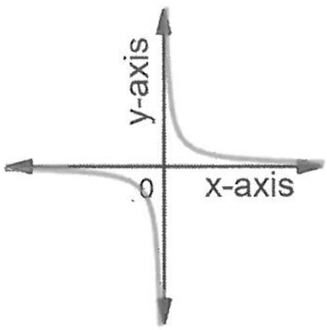
Square Root: $f(x) = \sqrt{x}$



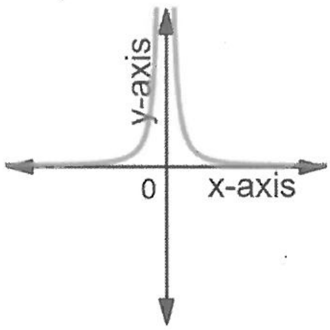
Cube Root: $f(x) = \sqrt[3]{x}$



Reciprocal: $f(x) = 1/x$



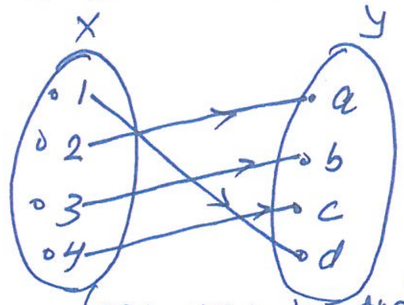
Reciprocal Squared: $f(x) = 1/x^2$



Types of functions

1) **one-one or injective functions :**

A function $f : X \to Y$ is defined to be one-one (or injective), if different element in X have different images in Y.
i.e. if no two different elements in X have the same image in Y.
In other words, each x in the domain has exactly one image in the range. And, no y in the range is the image of more than one x in the domain.



(one - one function)

Example :

a) Consider a function $f : R \to R$ is defined by $f(x) = 2x+3$.

let $f(x_1) = f(x_2)$

$2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$ Hence, f is one to one function.

b) If a function $f : R \to R$ is defined by $f(x) = |x| + x$

$f(-2) = |-2| + (-2) = 0$

$f(-3) = |-3| + (-3) = 0$

$f(-2) = f(-3)$ but $-2 \neq -3$

Hence, f is not one-one function.

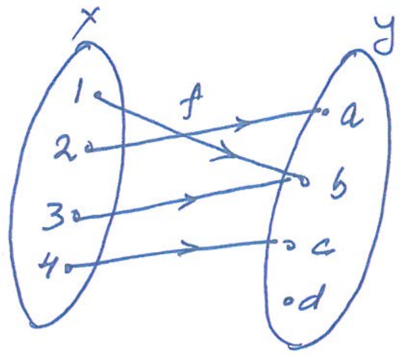
In general,
If for every $x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
or equivalently $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Injective means we won't have two or more "x" pointing to the same "y"
We can have "y" without matching 'x'.

2) Many-one function

A function which is not a one-one function is many-one function.

For example, consider a function $f(x) = x^2 \ \forall x \in \mathbb{Z}$.



Many-one function

Here $1 \neq -1$, but $f(1) = f(-1) = 1$

$2 \neq -2$, but $f(2) = f(-2) = 4$.

$f(x) = f(-x) = x^2$ is a many-one function.

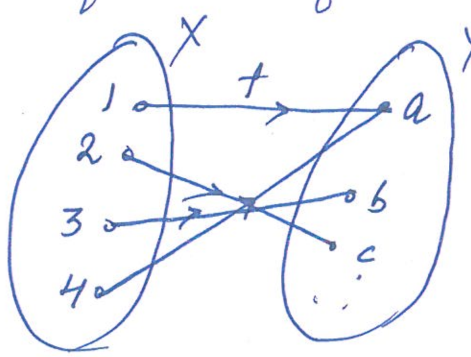
Onto function or Surjective function:

A function $f: X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f.

For every $y \in Y$, there exists an element x in X such that $f(x) = y$, $f: X \to Y$ is said to be onto (or surjective) function.

In onto function, Range of f = Co-domain of f.

Remarks: $f: X \to Y$ is an onto function if and only if range of $f = Y$.

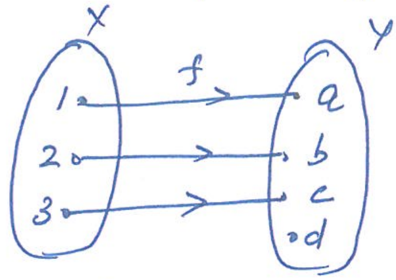


Surjective means that every "y" has at least one matching "x" (may be more than one). There won't be a "y" left out.

(3) Into function

A function $f: X \to Y$ is called into function if there is at least one element of Y which has no pre-image in X.

For example, $X = \{1, 2, 3\}$ $Y = \{a, b, c, d\}$



into function.

$f = \{(1,a), (2,b), (3,c)\}$

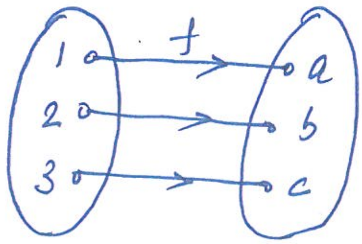
then $f: X \to Y$ is onto function.

4) Bijective function

A function $f: X \to Y$ is said to be bijective, if f is both one-one and onto.

Bijective means both injective and Surjective together.

For example, $X = \{1, 2, 3\}$
$Y = \{a, b, c\}$



$f = \{(1,a), (2,b), (3,c)\}$

then $f: X \to Y$ is both one-one and onto function (bijection).

Summary:

1) (i) Into Function: A function $f: X \to Y$ is called into function, if there exist at least one element in Y which has no pre-image in X.

(ii) A function $f: X \to Y$ is defined to be onto, which is also called surjective function, if every element of Y is the image of some element of X under f, i.e. for every $y \in Y$ there exists an element $x \in X$ such that $f(x) = y$.

(iii) Note that a function will either be into or onto.

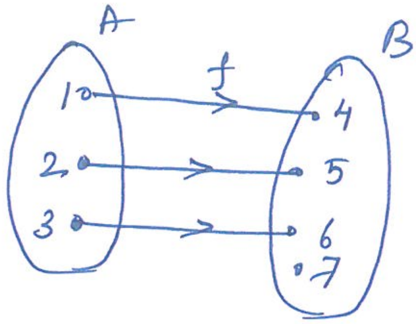
2) (i) One-one into functions: A function $f: X \to Y$ is said to be one-one into function if different element in X have different f image in Y and there exist an element in Y having no image X.

(ii) One-one onto function (Bijection): A function $f: X \to Y$ is said to be one-one onto if different element X have different f images in Y and there is no element in Y having no pre image in X.

(iii) A function which both one-one and onto is called bijective

Solved Examples

(1) Let $A = \{1, 2, 3\}$ $B = \{4, 5, 6, 7\}$ and let $f = \{(1,4), (2,5), (3,6)\}$ be a function from A to B. Show that f is one-one and into.



Soln:

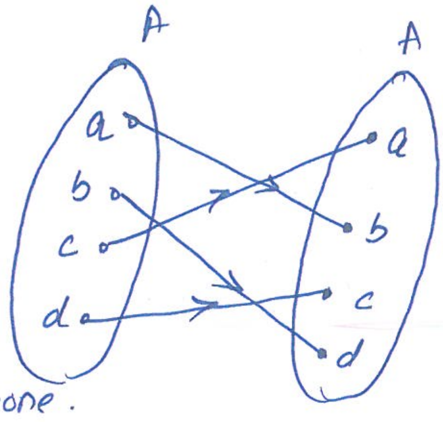
$f = \{(1,4), (2,5), 3, 6)\}$

$f(1) = 4$, $f(2) = 5$ $f(3) = 6$

Here we see that images of distinct element 1, 2, and 3 of A under f are 4, 5 and 6 respectively in B which are distinct $\therefore$ f is one-one.

Also, there is one element 7 of B which is not the f-image of any element of A. So range $\{4, 5, 6\} \neq$ co-domain of B. Hence, f is one-one and into function A to B.

2) If $A = \{a, b, c, d\}$ and f corresponds to the Cartesian product $\{(a,b), (b,d), (c,a), (d,c)\}$, show that f is one to one from A to A.



Soln: We have $f(a) = b$, $f(b) = d$
$f(c) = a$, $f(d) = c$

Since distinct elements a, b, c, d of A have distinct f-images b, d, a, c in A. Therefore f is one-one.

Again $f(A) = \{b, d, a, c\} = A$
$\therefore$ f is onto function.

Hence, f is one-to-one from A onto A.

3) Show that the function $f: N \to N$, given by $f(x) = 2x$ is one-one but not onto.

Soln: For every x_1 & x_2 $f(x) = 2x$

consider $f(x_1) = 2x_1$ $f(x_2) = 2x_2$

If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore f$ is one-one

If we consider 1 in the co-domain N,

i.e. $2x = 1 \Rightarrow x = \frac{1}{2} \notin N$.

There does not exist any $x \in N$ (domain) such that $f(x) = 1$. Thus $1 \in N$ has no pre-image in N.

$\therefore f$ is not onto.

Hence, $f(x) = 2x$ is one-one but not onto.

4) Prove that the function $f: R \to R$, given by $f(x) = 2x$ is one-one and onto.

Soln: For every x_1 & $x_2 \in R$ $f(x) = 2x$

consider $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

Again for any real number $y \in R$ (co-domain), there exists $\frac{y}{2}$ in R (domain), $f\left(\frac{y}{2}\right) = 2\left(\frac{y}{2}\right) = y$

$\therefore f$ is onto.

Hence, f is one-one and onto.

5) Show that function $f: R \to R$ defined by $f(x) = \frac{1}{x}$ is one-one and onto where R is the set of all non zero real numbers. Is the result true if the domain R is replaced by N with co-domain being same as R?

Soln: If we take two non-zero real numbers x_1 & x_2 then their images are $f(x_1)$ and $f(x_2)$ respectively.

Since $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

i.e. no element in the co-domain is the image of more than on element in the domain. Hence, f is one-one.

Again $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Now if y is any non-zero real number then $x = \frac{1}{y}$ is always real number i.e. each y in R has its pre-image in R. $\therefore$ f is onto.

Thus the function f is one-one as well as onto.
Hence, f is one-one onto.

In $f: N \to R$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\therefore f: N \to R$ is one-one.

Now, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

there exist y (say 2) $\in R$ such that $f(x) = \frac{1}{2} \notin N$

$\therefore$ f is not onto.

6) Check the injectivity and surjectivity of following function:

(i) $f: N \to N$ given by $f(x) = x^2$
(ii) $f: Z \to Z$ given by $f(x) = x^2$
(iii) $f: R \to R$ given by $f(x) = x^2$
(iv) $f: N \to N$ given by $f(x) = x^3$
(v) $f: Z \to Z$ given by $f(x) = x^3$

Sol^n: (i) $f: N \to N$, $f(x) = x^2$ let $x_1, x_2 \in N$
such that $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$, $x_1 = x_2$
$\therefore$ f is one-one (or injective).

Consider $3 \in N$ (co-domain) $x^2 = 3$, $x = \pm\sqrt{3} \notin N$
$\therefore$ there exist no natural number x such that $f(x) = 3$.
$\therefore$ f is not onto (surjective).
Hence, f is injective (one-one) but not surjective (not onto).

(ii) $f: Z \to Z$, $f(x) = x^2$
consider $x_1 = 1$ $x_2 = -1$ $f(x_1) = (1)^2 = 1$ $f(x_2) = (-1)^2 = 1$
$f(x_1) = f(x_2)$ but $x_1 \neq x_2$ $\therefore$ f is not one-one.

Consider $5 \in$ co-domain of Z then $f(x) = x^2 = 5$ $x = \pm\sqrt{5} \notin Z$.
$\therefore$ there exist no x in the domain of Z such that $f(x) = x^2 = 5$.
$\therefore$ f is not onto (surjective)
$\therefore$ f is neither injective nor surjective.

(iii) $f: R \to R$ $f(x) = x^2$. f is neither injective nor surjective

(iv) $f: N \to N$ $f(x) = x^3$.
For $x_1, x_2 \in N$ consider $f(x_1) = f(x_2)$ $x_1^3 = x_2^3$
$x_1^3 - x_2^3 = 0$; $(x_1 - x_2)\underbrace{(x_1^2 + x_1x_2 + x_2^2)}_{\text{can't be zero.}} = 0 \Rightarrow \underline{x_1 = x_2}$
$\therefore$ f is one-one

Since $2 \in N$ but there is no natural number x such that $f(x) = 2$ i.e. $x^3 = 2$. $\therefore$ f is not onto (or surjective.
Hence f is injective but not surjective.

(v) $f: Z \to Z$ $f(x) = x^3$. f is injective (one-on) but not surjective.

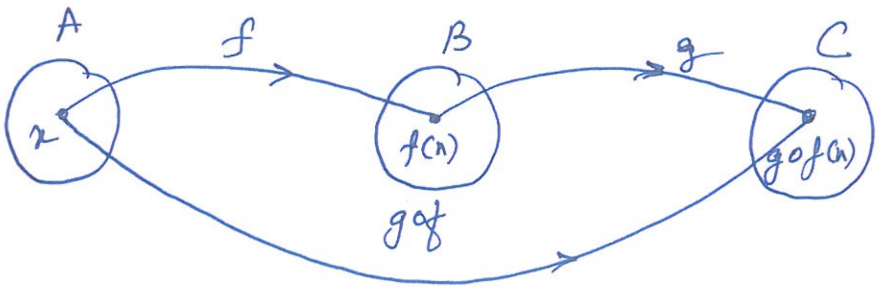
Composition of Functions

We have studied that functions can be combined using addition, subtraction, multiplication and division to create new functions. Another method for combining functions is to form the composition of one with the other.

Definition:

Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)), \ \forall \ x \in A.$$



Notes:

1) Notation gof means that the function f is applied first and then g is applied second.

2) The composition of f with g, fog is general not the same function as the composition of g with f gof. Furthermore, the functions may not have the same domain.

Examples

1) Let $f : R \to R$ be given by $f(x) = \sin x$ for all $x \in R$ and $g = R \to R$ be given by $g(x) = x^2$, for all $x \in R$.

The composite function $(gof) : R \to$ is given by $(gof)(x) = g(f(x)) = g(\sin x) = (\sin x)^2 = \sin^2 x \quad x \in R.$

2) Let $f: R \to R$ be given by $f(x) = 2x^2 - 3$ for all $x \in R$ and $g: R \to R$ be given by $g(x) = 4x$.

(i) $(f \circ g)(2) = f(g(2)) = f(8) = 2(8)^2 - 3 = 125$

(ii) $(g \circ f)(2) = g(f(2)) = g(2 \cdot 2^2 - 3) = g(5) = 5 \times 4 = 20$.

Theorem 1: If $f: A \to B$ and $g: B \to C$ are one-one, then $gof: A \to C$ is also one-one.

Theorem 2: If $f: A \to B$ and $g: B \to C$ are onto, then $gof: A \to C$ is also onto.

Theorem 3: If $f: X \to Y$, $g: Y \to Z$ and $h: Z \to X$ are functions, then $ho(gof) = (hog)of$.

Remarks:

1) If $f: A \to B$, $g: B \to C$ are two one-one onto functions then $gof: A \to C$ is also one-one onto.

2) It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

3) If $f: X \to Y$ is a function such that there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$ then f must be one-one and onto.

Properties of Composition of Functions :

1) Given two functions g and f, the composition of g and f, denoted by gof is defined by $(gof)(x) = g(f(x))$

$$gof \neq fog$$

i.e. the composition of function is not commutative.

2) If $f: X \to Y$, $g: Y \to Z$ and $h: Z \to X$ are functions, then $ho(gof) = (hog)of$ i.e. the composition of function of function is associative.

3) If f and g are two bijections, then gof is also a bijection i.e. the composition of two bijections is a bijection.

4) If $f: A \to B$ and I_A, I_B are identity function on A, B respectively, then $foI_A = I_Bof = f$ i.e. the composition of any function with the identity function is the function itself.

Invertible Functions

(i) A function $f: X \to Y$ is said to be invertible if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$

Function g is called the inverse of f and is denoted by f^{-1}.

So, $g = f^{-1}$.

f and g are bijections.

(ii) If f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

(iii) Inverse functions, if it exists, is unique.

(iv) Procedure to find inverse of a function.

Let $f: X \to Y$ be a bijection. In order to find $f^{-1}: Y \to X$ Put $y = f(x)$, where $x \in X$, $y \in Y$. Then solve $y = f(x)$ and find the value of x in terms of y. Replace x by $f^{-1}(y)$.

Solved Examples

1) If $f: R \to R$ and $g: R \to R$ are defined respectively as $f(x) = x^2 + 3x + 1$ $g(x) = 2x - 3$, find (i) fog (ii) gof

Soln: $f(x) = x^2 + 3x + 1$ $g(x) = 2x - 3$

(i) $fog(x) = f(g(x)) = f(2x-3)$

$= (2x-3)^2 + 3(2x-3) + 1$

$= [4x^2 + 9 - 12x] + 6x - 9 + 1$

$= 4x^2 + 9 - 12x + 6x - 9 + 1$

$= 4x^2 - 6x + 1$

(ii) $gof(x) = g(f(x)) = g(x^2 + 3x + 1) = 2(x^2 + 3x + 1) - 3$

$= 2x^2 + 6x - 1$

2) If $f: R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.

Soln: $f(x) = (3 - x^3)^{1/3}$

$fof(x) = f(f(x)) = f((3 - x^3)^{1/3})$

$= \left[3 - \{(3 - x^3)^{1/3}\}^3\right]^{\frac{1}{3}}$

$= [3 - (3 - x^3)]^{\frac{1}{3}}$

$= (\cancel{3} - \cancel{3} + x^3)^{1/3}$

$= x^{3 \times \frac{1}{3}}$

$fof(x) = x$

3) (i) If $f: R \to R$ is defined by $f(x) = 3x+2$ define $f(f(x))$.

(ii) Let $f = \{(1,2), (3,5), (4,1)\}$ $g = \{(2,3), (5,1), (1,3)\}$ then find gof and fog.

Soln (i) $fof(x) = f(f(x)) = f(3x+2) = 3(3x+2)+2 = 9x+8$

(ii) $f = \{(1,2), (3,5), (4,1)\}$ $g = \{(2,3), (5,1), (1,3)\}$

$gof(1) = g(f(1)) = g(2) = 3$

$gof(3) = g(f(3)) = g(5) = 1$

$gof(4) = g(f(4)) = g(1) = 3$

$\therefore gof = \{(1,3), (3,1), (4,3)\}$

$fog(2) = f(g(2)) = f(3) = 5$

$fog(5) = f(g(5)) = f(1) = 2$

$fog(1) = f(g(1)) = f(3) = 5$

$\therefore fog = \{(2,5), (5,2), (1,5)\}$

4) Find gof and fog, if (i) $f(x) = |x|$ and $g(x) = |5x-2|$
(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Soln: (i) $f(x) = |x|$ $g(x) = |5x-2|$

$gof(x) = g(f(x)) = g(|x|) = |5|x|-2|$

$fog(x) = f(gx) = f(|5x-2|) = ||5x-2|| = |5x-2|$.

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$gof(x) = g(f(x)) = g(8x^3) = [8x^3]^{\frac{1}{3}} = 2x$$

$$fog(x) = f(g(x)) = f(x^{\frac{1}{3}}) = 8(x^{\frac{1}{3}})^3 = 8x.$$

5) Let $Y = \{n^2 : n \in N \subset N\}$. Consider $f: N \rightarrow Y$ as $f(n) = n^2$. Show that f is invertible. Find the inverse of f.

<u>Soln</u> Let $n_1, n_2 \in N$ such that $f(n_1) = f(n_2)$

$f(n) = n^2$

$n_1^2 = n_2^2 \Rightarrow n_1 = \pm n_2$

as n_1 & $n_2 \in N$ $\therefore n_1 = n_2$

$\therefore f$ is one-one function.

Let y be an arbitrary element of co-domain Y, then $y = n^2$.

$\therefore f$ is onto

[$\because$ Range = Co-domain]

As f is <u>one-one and onto, so f is invertible</u>.

Again $y = n^2$ $n = \sqrt{y}$

[$\because$ n is a natural number so +ve square root being taken].

as $y = n^2 = f(n)$, so $n = f^{-1}(y)$

But $n = \sqrt{y}$

$\therefore f^{-1} : Y \rightarrow N$, defined by $f^{-1}(y) = \sqrt{y}$

changing y to x we get $f^{-1}(x) = \sqrt{x}$

6) If the function $f: R \to R$ be given by $f(x) = x^2+2$ and $g: R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$. find fog and gof and hence find $fog(2)$ and $gof(-3)$.

Soln: $f(x) = x^2+2 \qquad g(x) = \frac{x}{x-1} \qquad x \neq 1.$

$$fog(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2$$

$$= \frac{3x^2-4x+2}{(x-1)^2}$$

$$fog(2) = \frac{3(2)^2-4(2)+2}{(2-1)^2} = 6$$

$$gof(x) = g(f(x)) = g(x^2+2) = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1}$$

$$gof(-3) = \frac{(-3)^2+2}{(-3)^2+1} = \frac{11}{10}.$$

7) Let $f: R \to R$ be defined as $f(x) = 10x+7$. Find the function $g: R \to R$ such that $gof = fog = I_R$.

Soln: Let $f: X \to Y$ where X, Y are set of real numbers. Let y be any arbitrary elements of Y.

then $\quad y = f(x) = 10x+7 \quad x \in X$

This gives $\quad x = \frac{y-7}{10}$

Define $g: Y \to X$ by $g(y) = \frac{y-7}{10}$

$$gof(x) = g(f(x)) = g(10x+7) = \frac{10x+7-7}{10} = \frac{10x}{10} = x$$

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right)+7 = y$$

This shows that $gof = fog = I_R$.

8) Consider $f: R \to R$ given by $f(x) = 4x+3$. Show that f is invertible. Find the inverse of f.

Sol^n: $f: R \to R$ given by $f(x) = 4x+3$

Let x_1, x_2 be any arbitrary elements of the domain R of f such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 4x_1 + 3 = 4x_2 + 3$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ $f: R \to R$ is one-one function.

Let y be arbitrary element of co-domain of R of f.

then $y = f(x) = 4x+3 \quad x \in R$

$x = \dfrac{y-3}{4}$ which is real number for all $y \in R$.

$\therefore$ Every element y in the co-domain has its pre-image x in domain given by $x = \dfrac{y-3}{4}$

$\therefore$ f is onto function.

Since f is one-one and onto function, so f is invertible.

f^{-1} is given by $f^{-1}(y) = \dfrac{y-3}{4}$

Replacing y by x, $f^{-1}(x) = \dfrac{x-3}{4}$.

9) Show that $f: [-1,1] \to R$, given by $f(x) = \dfrac{x}{x+2}$ is one-one. Find the inverse of the function $f: [-1,1] \to$ Range of f.

Binary Operations

(i) A function $f: A \times A$ is called a binary operation $*$ on A and the set is said to be closed with respect to the operation.
In other words, a binary operation '$*$' on A is a rule which assign to a pair $a, b \in A$ another element $a*b \in A$.

(ii) If '$*$' is a binary operation on A, then $(A, *)$ denotes the set A with binary operation '$*$'.

Idempotent, Commutative and Associative Operations.

Particular binary operations may posses certain important properties, which are defined below:
Let A be any set. Then an operation $*$ on A is

1.) Idempotent if $a * a = a \ \forall a \in A$

2.) Commutative if $a * b = b * a \quad \forall a, b \in A$

3.) Associative if $(a*b)*c = a*(b*c) \ \forall a, b, c \in A$.

4.) Identity Element : consider $(A, *)$. If there exists $e \in A$ such that $a * e = e * a = a$, for all $a \in A$. Then e is called identity element of $*$ on set A.

5.) invertible element with respect to the operation $*$
An element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in set A such that $a * b = e = b * a$, b is called the inverse of a and is denoted by $\bar{a}'$.

Inverse of an Element

Suppose $(A, *)$ has an identity element e. Let $a \in A$. Then a is called an invertible element if there exists $b \in A$ such that $a*b = e = b*a$. The element b is called the inverse of a.

If $b, c \in A$ are inverse elements of $a \in A$, then $b = b*e = b*(a*c) = (b*a)*c = e*c = c$.

$\therefore$ The inverse of $a \in A$ is uniquely determined, if $(A, *)$ is associative and is denoted by a^{-1}.

If $(A, *)$ is associative and a is invertible, then $(a^{-1})^{-1} = a$.

Example : We can define operation $*$ on the set $A = \{0, 1, 2\}$ by the rule : if $x, y \in A$, then $x*y = \max[x, y]$

This operation is represented geometrically in figure.

*	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

Properties of binary operations :

1. Commutative property : A binary operation $*$ on the set X is called commutative, if $a*b = b*a$, for every $a, b \in X$.
2. Associative Property : A binary operation $* : A \times A \to A$ is said to be associative if $(a*b)*c = a*(b*c)$ $\forall a, b, c \in A$.

Solved Examples

1) (i) Let $*$ be a binary operation defined by $a*b = 2a+b-3$. Find $3*4$.

(ii) If the binary operation $*$ on the set of integers Z, is defined by $a*b = a+3b^2$, then find the value of $8*3$.

Sol^n: (i) $a*b = 2a+b-3$

$$3*4 = 2(3)+4-3$$
$$= 6+1$$
$$= 7$$

(ii) $a*b = a+3b^2$, $8*3 = 8+3(3)^2 = 8+27 = 35$.

2) (i) The binary operation $* \; R\times R \to R$ is defined as $a*b = 2a+b$ find $(2*3)*4$.

(ii) If $*$ be a binary operation on set of integers I, defined by $a*b = 3a+4b-2$, find the value of $4*5$.

Sol^n:

$$a*b = 2a+b$$
$$2*3 = 2\times2+3 = 4+3 = 7$$
$$(2*3)*4 = 2\times7+4 = 14+4 = 18.$$

(ii) $a*b = 3a+4b-2$

$$4*5 = 3\times4+4\times5-2 = 12+20-2 = 30$$

(3) Let $*$ be a binary operation on the set of rational numbers given as $a*b = (2a-b)^2$, $a, b \in Q$. Find $3*5$ and $5*3$. Is $3*5 = 5*3$?

Sol^n:

$$a*b = (2a-b)^2$$
$$3*5 = [2\times3-5]^2 = (6-5)^2 = 1$$
$$5*3 = [2\times5-3]^2 = [10-3]^2 = 49$$
$$\therefore 3*5 \neq 5*3.$$

(4) (i) Is the binary operation $*$ defined on set Q, given by $a*b = \frac{a+b}{2}$ for $a, b \in Q$ commutative?

(ii) Is the above binary operation $*$ associative?

Solⁿ: (i) $a*b = \frac{a+b}{2}$ $\quad b*a = \frac{b+a}{2}$

$= \frac{a+b}{2}$

$\therefore$ Binary operation $*$ is commutative.

(ii) $a*(b*c) = a*\left(\frac{b+c}{2}\right) = \frac{a+\frac{b+c}{2}}{2} = \frac{2a+b+c}{4}$

$(a*b)*c = \frac{a+b}{2}*c$

$= \frac{\frac{a+b}{2}+c}{2} = \frac{a+b+2c}{4}$

$\therefore a*(b*c) \neq (a*b)*c$.

Hence, binary operation $*$ is not associative.

(5) Let $*$ be a binary operation on N given by $a*b = HCF(a,b)$, $a, b \in N$. Write down the value of $22*4$.

Solⁿ: $a*b = HCF(a,b)$

$22*4 = HCF(22,4)$

$= \underline{2}$

$$\begin{array}{r} 4)\overline{22}(5 \\ 20 \\ \hline 2)4 \\ 4 \\ \hline 0 \end{array}$$

6) Consider the infimum binary operation $\wedge$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min a$ and b. Write the composition table of the operation $\wedge$.

Solⁿ

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5