Introduction to Three Dimensional Geometry

Basics

Cartersian plane

- A Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length.

Coordinates of a Point

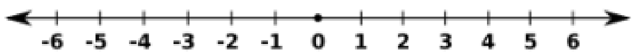
- The coordinates of a point are a pair of numbers that define its exact location on a two-dimensional plane
- A number on the x-axis called an x-coordinate, and a number on the y-axis called a y-coordinate.

Ordered Pair

- An ordered pair contains the coordinates of one point in the coordinate system.
- The order in which you write x- and y-coordinates in an ordered pair is very important.
- The x-coordinate always comes first, followed by the y-coordinate
- There is also a three coordinate called Z coordinate

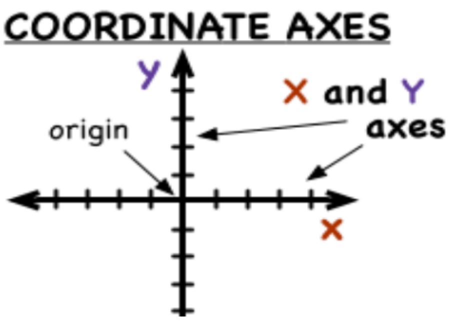
One-Dimensional Coordinate Systems

- A 1-dimensional coordinate system, also known as a number line, uses one coordinate to tell how far away from zero it is located.



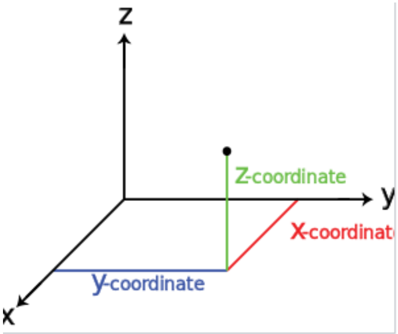
Two-Dimensional Coordinate Systems

- The below shown figure represens the **Co-ordinate plane** or it is also called as **Cartersian plane OR XY-Plane**
- Each reference line is called a **coordinate axis or just axis (plural axes)** of the system
- The coordinates are written as an ordered pair (x, y).
- The point where the axes meet is the common origin of the two number lines and is simply called the origin.
- It is often labelled O and if so then the axes are called Ox and Oy.
- The value of x is called the x-coordinate or abscissa
- The value of y is called the y-coordinate or ordinate



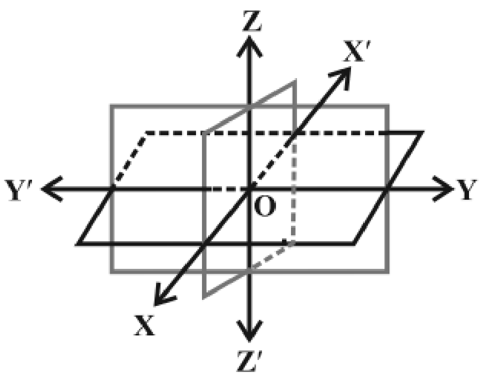
Three Dimensional Coordinate Axis

- The three straight lines having a common point that are the intersections of the three coordinate planes of reference in three-dimensional Cartesian geometry
- Three-dimensional space by means of three coordinates.
- Three coordinate axes are given, each perpendicular to the other two at the origin, the point at which they cross. They are usually labelled x, y, and z.
- The position of any point in three-dimensional space is given by an ordered triple of real numbers, each number giving the distance of that point from the origin measured along the given axis, which is equal to the distance of that point from the plane determined by the other two axes
- The coordinates are written as an ordered pair (x, y, z).



Coordinate Axes and Coordinate Planes in the Three Dimensional Space

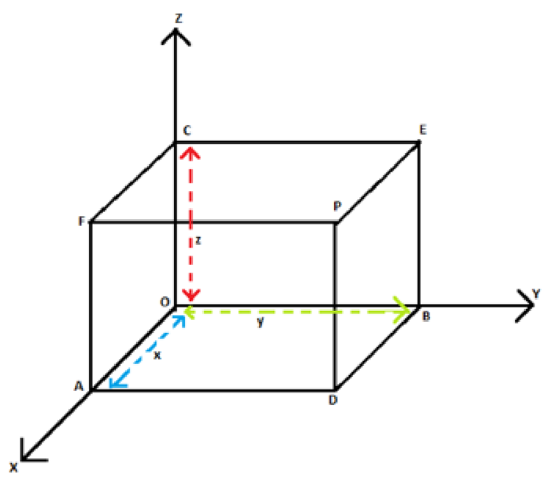
- Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other
- Three planes intersect along the lines X'OX, Y'OY and Z'OZ, called the x, y and z-axes
- The planes XOY, YOZ and ZOX, called, respectively the XY-plane, YZ-plane and the ZX-plane, are known as the three coordinate planes.
- The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative.
- The distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY′ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX′ as negative.
- The point O is called the origin of the coordinate system.
- The three coordinate planes divide the space into eight parts known as octants. These octants could be named as XOYZ, X'OYZ, X'OY'Z, XOY'Z, XOYZ′, X'OYZ′, X'OY'Z′ and XOY'Z′. and denoted by I, II, III, ..., VIII , respectively



Coordinates of a Point in Space

- Choose a rectangular coordinate system.
- Let P be a point with coordinates by an ordered 3-tuple (x, y, z).

- Drop a perpendicular PB on the XY plane and then from the point B, drop perpendiculars BA and BC on x – axis and y – axis respectively.



- Assume the length of the perpendiculars BC, BA and PB as x, y and z respectively. These lengths x, y and z are known as the co-ordinates of the point P in three dimensional space.
- The co-ordinates of the origin O is (0 , 0 , 0)
- The co-ordinates of the point A is given by (x , 0 , 0)as A lies completely on the x – axis.
- The co-ordinates of any point on y – axis is given as (0 , y , 0) and on z – axis the coordinates are given as (0 , 0 , z).
- The co-ordinates of any point in three planes XY , YZ and ZX will be (x,y,0) , (0,y,z) and (x,0,z) respectively.
- To locate a point,i.e. when the co-ordinates of the point are given
    - We have to draw three planes parallel to XY , YZ and ZX plane meeting the three axes in points A,B and C
    - Let OA = x, OB = y and OC = z.
    - The co-ordinates of the point are given as (x, y, and z).
    - The planes ADPF, BDPE and CEPF intersect at point P which corresponds to the ordered triplet (x, y, z).
- To determine the octant in which a point lies, the signs of the co-ordinates of a point in the table below are used

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Note:

- While giving the co-ordinates of a point, we always write them in order such that the coordinate of x – axis comes first, followed by the co-ordinate of y-axis and then z – axis. Thus for each point in space there exists an ordered 3-tupleof real numbers for its representation

Example

1. In the below figure, if P is (2, 4, 5), find the coordinates of F.

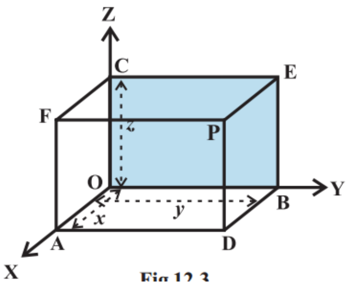


Fig 12.3

Solution:

For the point F, the distance measured along OY is zero. Therefore, the coordinates of F are (2, 0, 5)

2. Find the octant in which the points (–3, 1, 2) and (–3, 1, – 2) lie.

Solution

From the Table above in the previous page, the point (–3, 1, 2) lies in second octant and the point (–3, 1, – 2) lies in octant VI.

Distance between Two Points

- Let P(x1, y1, z1) and Q (x2, y2, z2) be two points referred to a system of rectangular axes OX, OY and OZ.
- Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelepiped with one diagonal PQ

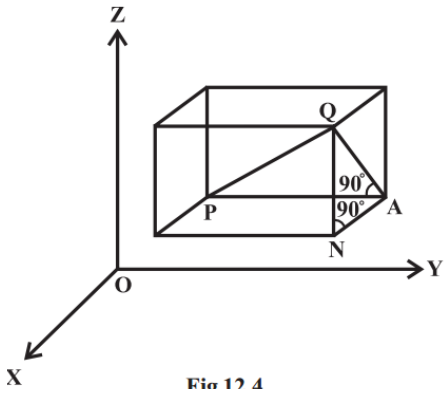


Fig 12.4

- Now, since ∠PAQ is a right angle, in triangle PAQ,
  - $PQ^2 = PA^2 + AQ^2$ ... (1)
- Also, triangle ANQ is right angle triangle with ∠ANQ a right angle.
- Therefore $AQ^2 = AN^2 + NQ^2$... (2)
- From (1) and (2), we have
  - $PQ^2 = PA^2 + AN^2 + NQ^2$
- Now $PA = y_2 – y_1$, $AN = x_2 – x_1$ and $NQ = z_2 – z_1$
- Hence $PQ^2 = (x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2$
- Therefore $PQ = \sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$
- This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).
- In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{(x_2^2+ y_2^2+z_2^2)}$, which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Example:

Find the distance between the points P (1, –3, 4) and Q (– 4, 1, 2).

Solution:

The distance PQ between the points P (1,–3, 4) and Q (– 4, 1, 2) is

$$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2}$$
$$= \sqrt{25+16+4}$$
$$= \sqrt{45} = 3\sqrt{5} \text{ units}$$

Note:

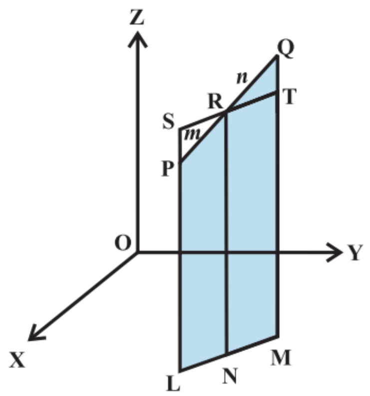
- If three points P,Q,R are collinear, then sum of two points is equal to the third point
- **Example**
    - Show that the points P (–2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear.
        - Solution
        - We know that points are said to be collinear if they lie on a line.

$$PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$$
$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$$
$$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$$

PQ + QR = PR. Hence, P, Q and Rare collinear.

Section Formula

- Let the two given points be P(x1, y1, z1) and Q (x2, y2, z2). Let the point R (x, y, z) divide PQ in the given ratio m: n internally.
- Draw PL, QM and RN perpendicular to the XY-plane.



- PL | | RN | | QM and feet of these perpendiculars lie in a XY-plane.
- The points L, M, N will lie on a line which is the intersection of the plane containing PL, RN and QM with the XY-plane.
- Through the point R draw a line ST parallel to the line LM.
- Line ST will intersect the line LP externally at the point S and the line MQ at T

- Also Quadrilaterals LNRS and NMTR are parallelograms.
- The triangles PSR and QTR are similar.
- Therefore,

$$\frac{m}{n} = \frac{\mathrm{PR}}{\mathrm{QR}} = \frac{\mathrm{SP}}{\mathrm{QT}} = \frac{\mathrm{SL}-\mathrm{PL}}{\mathrm{QM}-\mathrm{TM}} = \frac{\mathrm{NR}-\mathrm{PL}}{\mathrm{QM}-\mathrm{NR}} = \frac{z-z_1}{z_2-z}$$

This implies $$z=\frac{mz_2+nz_1}{m+n}$$

- Similarly, by drawing perpendiculars to the XZ and YZ-planes, we get

$$y = \frac{my_2 + ny_1}{m+n} \text{ and } x = \frac{mx_2 + nx_1}{m+n}$$

- Hence, the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m : n are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

- If the point R divides PQ externally in the ratio m : n, then its coordinates are obtained by replacing n by – n so that coordinates of point R will be

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Case 1

- Coordinates of the mid-point: In case R is the mid-point of PQ, then m : n = 1 : 1 so that $x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}$ and $z = \frac{z_1 + z_2}{2}$
- These are the coordinates of the midpoint of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2)

Case 2

- The coordinates of the point R which divides PQ in the ratio k : 1 are obtained by taking k mn = which are as given below:

$$\left(\frac{kx_2+x_1}{1+k}, \frac{ky_2+y_1}{1+k}, \frac{kz_2+z_1}{1+k}\right)$$

Example:

Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2: 3 (i) internally, and (ii) externally.

Solution

(i) Let P (x, y, z) be the point which divides line segment joining A (1, – 2, 3) and B (3, 4, –5) internally in the ratio 2: 3. Therefore

$$x = \frac{2(3) + 3(1)}{2+3} = \frac{9}{5},\ y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5}\quad,\ z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5},\frac{2}{5},\frac{-1}{5}\right)$

(ii) Let P (x, y, z) be the point which divides segment joining A (1, –2, 3) and B (3, 4, –5) externally in the ratio 2: 3. Then

$$x = \frac{2(3)+(-3)(1)}{2+(-3)} = -3,\ y = \frac{2(4)+(-3)(-2)}{2+(-3)} = -14,\ z = \frac{2(-5)+(-3)(3)}{2+(-3)} = 19$$

Therefore, the required point is (–3, –14, 19).

Coordinates of a Triangle

- Let ABC be the triangle.
- Let the coordinates of the vertices A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), respectively. Let
- D be the mid-point of BC.
- Hence coordinates of D are

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2}\right)$$

- Let G be the centroid of the triangle. Therefore, it divides the median AD in the ratio 2: 1. Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1},\frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1},\frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$