Chap 5 : Complex Numbers and Quadratic Equation (XI)

Introduction

We have seen that the equation $x^2+1=0$ has no real solution as $x^2+1=0$ gives $x^2=-1$. The main objective is to solve the equation $ax^2+bx+c=0$, where $D = b^2-4ac < 0$, which is not possible in the system of real numbers. Euler was the first Mathematician to introduce the symbol i, that is i for $\sqrt{-1}$ with the property $i^2=-1$.

Imaginary Numbers.

Square root of a negative number is called an imaginary number. For example $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-17}$ etc.
$\sqrt{-1}$ is called an imaginary unit iota, which is denoted by i. Therefore, $i=\sqrt{-1}$.

Integral Powers of i : $i=\sqrt{-1}$, $i^2 = i\cdot i = \sqrt{-1}\times\sqrt{-1} = -1$

$i^3 = i^2\cdot i = (-1)i = -i$ $\quad$ $i^4 = (i^2)^2 = (-1)^2 = 1$ so on.

$$i^{-1} = \frac{1}{i} = \frac{i^4}{i} = i^3 = -i \qquad i^{-2} = \frac{1}{i^2} = \frac{i^4}{i^2} = i^2 = -1$$

$i^0 = 1$. $\quad$ $i^n = -1, 1, i$ or $-i$

$$\sqrt{-a}\times\sqrt{-b} = i\sqrt{a}\times i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}$$

$i = \sqrt{-1}$ $\qquad$ $i^{-1} = -i$

$i^2 = -1$ $\qquad$ $i^{-2} = -1$

$i^3 = -i$ $\qquad$ $i^{-3} = i$

$i^4 = 1$ $\qquad$ $i^{-4} = 1$

$i = \sqrt{-1}$ $\quad i^2 = -1$ $\quad i^{-1} = -i$ $\quad i^{-3} = i$

$i^3 = -i$ $\quad i^4 = 1$ $\quad i^{-2} = -1$ $\quad i^{-4} = 1$

$i^2 = i^{-2} = -1$

$i^4 = i^{-4} = 1$

Example-1 : Evaluate the following :

(i) i^9 (ii) i^{342} (iii) i^{998} (iv) i^{-63} (v) $\left(i^3 + \frac{1}{i^3}\right)$

Solution : (i) $i^9 = i^8 \cdot i = (i^4)^2 \cdot i = (1)^2 \cdot i = i$ $\quad [\because i^4 = 1]$

(ii) $4)\,342\,(85$ $\quad Q = 85$
$\quad 32$
$\quad 22$
$\quad 20$
$\quad 2$ $\quad R = 2$

$\therefore i^{342} = (i^4)^{85} \times i^2$ $\quad [\because i^4 = 1$
$= (1)^{85} \times (-1)$ $\quad i^2 = -1\,]$
$= -1$

(iii) $4)\,998\,(249$ $\quad Q = 249$
$\quad 8$
$\quad 19$
$\quad 16$
$\quad 38$
$\quad 36$
$\quad 2$ $\quad R = 2$

$\therefore i^{998} = (i^4)^{249} \times i^2$
$= (1)^{249} \times -1$
$= -1$

(iv) $i^{-63} = \dfrac{1}{i^{63}} = \dfrac{i}{i^{64}} = \dfrac{i}{(i^4)^{16}} = i$

(v) $i^3 + \dfrac{1}{i^3} = -i + \dfrac{1}{-i} = -i - \dfrac{1}{i} = -\left(i + \dfrac{1}{i}\right)$

$= -(i + i^{-1}) = -(i + (-i))$ $\quad [\because i^{-1} = -i]$

$= 0$

Example 2:

Evaluate the following: (i) $\sqrt{-25} \times \sqrt{-81}$ (ii) $\sqrt{-36} \times \sqrt{16}$

(iii) $4\sqrt{-4} \times 5\sqrt{-9} - 3\sqrt{-16}$

(i) $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$ $\quad \sqrt{-25} \times \sqrt{-81} = \sqrt{25}\,i \times \sqrt{81}\,i = 5i \times 9i$

$= 45i^2 = -45$

(ii) $\sqrt{-36} \times \sqrt{16} = 6i \times 4 = 24i$

(iii) $4\sqrt{-4} + 5\sqrt{-9} - 3\sqrt{-16} = 4 \times 2 \times i + 5 \times 3 \times i - 3 \times 4 \times i$

$= 8i + 15i - 12i$

$= 15i - 4i$

$= 11i$

Example 3: Prove that $\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2 = 2i$

$$\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2 = \left[i^{16} \times i - \frac{1}{(i^4)^8 \times i^2}\right]^2$$

$4)34(8$
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2

$$= \left(i - \frac{1}{i^2}\right)^2 = \left\{i - \frac{1}{(-1)}\right\}^2 = (i+1)^2$$

$= i^2 + 2i + 1$

$= \cancel{-1} + 2i + \cancel{1}$

$= 2i$

Example 4: for any positive integer n, prove that: $i^n + i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} + i^{n+5} + i^{n+6} + i^{n+7} = 0$

Sol^n: LHS $= i^n + i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} + i^{n+5} + i^{n+6} + i^{n+7}$

$= i^n(1 + i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7)$

$= i^n[(1 + i + (-1) + (-i) + 1 + i + (-1) + (-i)]$

$= i^n[\cancel{1} + \cancel{i} - \cancel{1} - \cancel{i} + \cancel{1} + \cancel{i} - \cancel{1} - \cancel{i}]$

$= i^n \times 0$

$= 0$

$=$ RHS.

Example 4: Proove that :

(i) $1 + i^{10} + i^{100} - i^{1000} = 0$

(ii) $i^{107} + i^{112} + i^{117} + i^{122} = 0$

(iii) $(1 + i^{14} + i^{18} + i^{22})$ is a real number.

Example 5: Evaluate the following

(i) $\sqrt{-81}$ Ans $9i$ (ii) $(\sqrt{-2})^6$ Ans -8

(iii) $\sqrt{-25} \times \sqrt{-49}$ Ans : -35 (iv) $\frac{2}{3} \times \sqrt{\frac{-9}{16}}$ Ans : $\frac{1}{2} i$

(v) $\sqrt{\frac{-49}{25}} \times \sqrt{\frac{-1}{9}} = -\frac{7}{15}$ (vi) $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$ Ans $= 0$

Example 6: Simplify : $\sqrt{\frac{-x}{16}} + \sqrt{\frac{-x}{25}} - \sqrt{\frac{-x}{36}}$ Ans : $\frac{17}{60} i\sqrt{x}$

: where 'x' is a positive real number

Example 7: Prove that : $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

Example 8 : Proove that $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25} = 1 + 4i$

Example 9 : Proove that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

Example 10 : Show that $(-\sqrt{-1})^{4n+3} = i$

where n is a positive integer.

COMPLEX NUMBER

A number of the form $x+iy$, where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number.

$$z = x+iy$$

x is called real part of z

y is called imaginary part of z.

Example : $3+5i$, $\sqrt{2}-3i$, $\frac{1}{3}+2i$, $-1+i\sqrt{3}$ etc.

$z = 2+i5$, then $\text{Re}\, z = 2$ and $\text{Im}\, z = 5$.

$z = iy$ is called purely imaginary

$z = x$ is called purely real.

Thus complex numbers contain real and imaginary numbers.

Equality of Two complex Numbers.

Two complex numbers are said to be equal if and only if their real as well as imaginary parts are equal.

If x_1+iy_1 and x_2+iy_2 are two complex numbers, then:

$$x_1+iy_1 = x_2+iy_2 \iff x_1 = x_2 \text{ and } y_1 = y_2$$

Example: If $4x + i(3x-y) = 3+i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution : $4x + i(3x-y) = 3+i(-6)$

Equating the real and imaginary parts

$4x = 3$ $\qquad$ $3x-y = -6$

$x = \frac{3}{4}$ $\qquad$ $3\times\frac{3}{4} - y = -6$ $\qquad$ $y = \frac{33}{4}$.

Algebra of Complex Numbers.

Addition of two complex numbers

If $z_1 = x_1 + iy_1$ $\quad z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

Negative of a Complex Numbers

If $z = x + iy$ is a complex number, then its negativ which is denoted by $-z$ is defined as:

$$-z = -x - iy$$

$-z$ is called the additive inverse of z.

Difference of Two Complex Numbers

If $z_1 = x_1 + iy_1$ $\quad z_2 = x_2 + iy_2$

then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

The following properties hold under addition

(1) The closure law: The sum of any two complex numbers is a complex number.

(2) The commutative law: $z_1 + z_2 = z_2 + z_1$

(3) The associative law: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(4) Existence of Identity: The complex number $0 + 0i$ is called additive identity or the zero complex number, $z + 0 = z$.

(5) Existence of Inverse: If z is complex number then $-z$ is its additive inverse. $z + (-z) = 0$.

Multiplication of Two Complex Numbers.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

The following properties hold under multiplication :

(1) Closure : The product of two complex number is a complex number.

(2) Commutativity : $z_1 z_2 = z_2 z_1$

(3) Associativity : $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$

(4) Existence of Identity : The complex number $1 + i0$, is called the multiplicative identity such that $z \cdot 1 = z$.

(5) Existence of Inverse : If z is a complex number, then we have a complex number z', or $\frac{1}{z}$ or z^{-1} such that
$$z \cdot z' = 1$$
$$z' = \frac{1}{z} = z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

(6) Distributive Law : (a) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

Conjugate of a complex number.

When two complex numbers differ only in the sign of i, they are said to be conjugates of each other.
Thus $x+iy$ and $x-iy$ are two conjugate complex numbers.

The conjugate of a complex number z is denoted by $\bar{z}$

$z = x+iy \qquad \bar{z} = x-iy$.

Properties of Conjugates :

(1) The conjugate of the conjugate of a complex number is complex number itself. $\overline{(\bar{z})} = z$.

(2) The sum and product of two conjugate complex number are purely real.
$z = x+iy \qquad \bar{z} = x-iy$
(i) Sum $\quad z+\bar{z} = x+iy+x-iy = 2x$, purely real

(ii) Product $\quad z\cdot\bar{z} = (x+iy)(x-iy) = x^2+y^2$, purely real.

(3) The conjugate of the sum (product) of two complex numbers is sum (product) of their conjugate.
$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1\cdot\bar{z}_2$

(4) (i) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$ $\qquad$ (ii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Modulus of a Complex Number (Absolute Value)

If $z = x+iy$ then $|z| = \sqrt{x^2+y^2}$ is called its Modulus.

$$z \cdot \bar{z} = |z|^2 \qquad z = x+iy \qquad \bar{z} = x-iy$$

Properties:

(1) for any complex number z, $|z| = 0 \iff z = 0$

(2) $|z| = |\bar{z}| = |-z|$

(3) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

(4) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(5) (i) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re\,(z_1 \cdot \bar{z}_2)$

(ii) $|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,Re\,(z_1 \cdot \bar{z}_2)$

(iii) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(6) Triangle Inequality. $|z_1+z_2| \leq |z_1| + |z_2|$

(7) $|z_1-z_2| \geq |z_1| - |z_2|$

(8) $|z_1-z_2| \geq \big||z_1| - |z_2|\big|$

Square Roots of a Complex Number

Let $x+iy$ be a square root of the complex number $a+ib$
then $\sqrt{a+ib} = x+iy$ $(x, y \in R)$

$$(x+iy)^2 = a+ib \Rightarrow x^2-y^2+i(2xy) = a+ib$$

Equating real and imaginary parts, we get :

$$x^2-y^2 = a \qquad 2xy = b$$

Now, $(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 = a^2+b^2$

$x^2+y^2 = \pm\sqrt{a^2+b^2}$ Since $x, y \in R$, x^2+y^2 is positive.

$\therefore x^2+y^2 = \sqrt{a^2+b^2}$

and $x^2-y^2 = a$

Adding, $2x^2 = \sqrt{a^2+b^2}+a$

$x^2 = \frac{1}{2}(\sqrt{a^2+b^2}+a) = h$ (say)

Subtracting, $2y^2 = \sqrt{a^2+b^2}-a$

$y^2 = \frac{1}{2}(\sqrt{a^2+b^2}-a) = k$ (say)

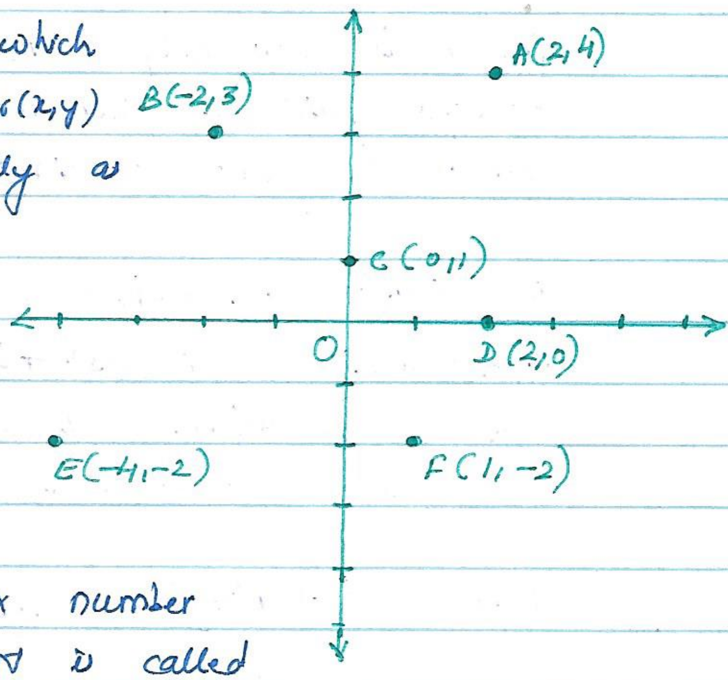
Thus, $x = \sqrt{h}, -\sqrt{h}$ and $y = \sqrt{k}, -\sqrt{k}$

Case I : When b is +ve, $x+iy = \sqrt{h}+i\sqrt{k}$
$x+iy = -\sqrt{h}-i\sqrt{k}$

Case II : When b is -ve, $x+iy = \sqrt{h}-i\sqrt{k}$
$x+iy = -\sqrt{h}+i\sqrt{k}$

Argand Plane (Complex plane)

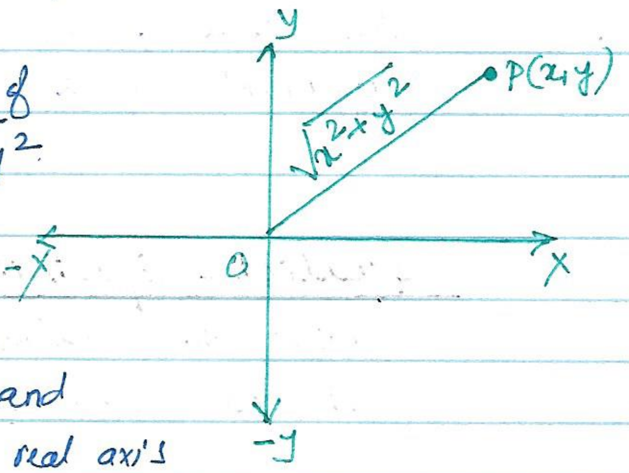
The complex number $x+iy$ which corresponds to the ordered pair (x,y) can be represented geometrically as unique point $P(x,y)$ in the XY-plane and vice-versa.



$2+4i$ corresponds to $A(2,4)$
$-2+3i$ corresponds to $B(-2,3)$
etc.

The plane having a complex number assigned to each of its point is called the complex plane or the Argand plane.

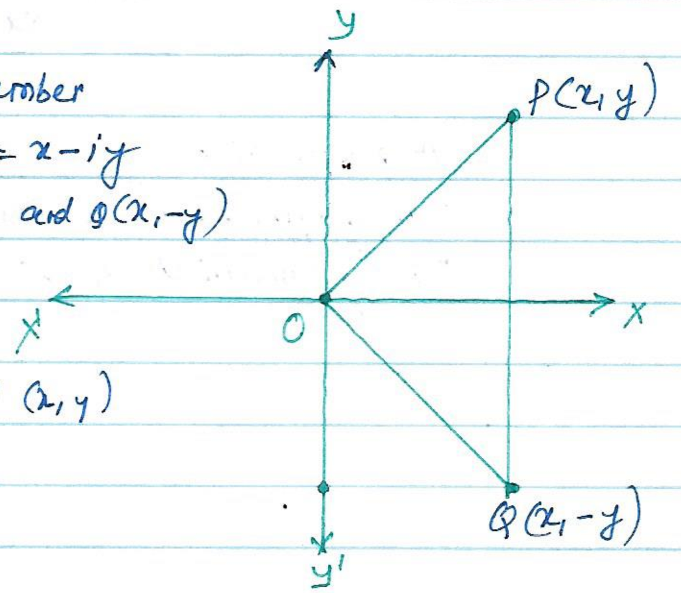
In the Argand plane, the modulus of the complex number $x+iy = \sqrt{x^2+y^2}$ is the distance between the point $P(x,y)$ to the origin $O(0,0)$.



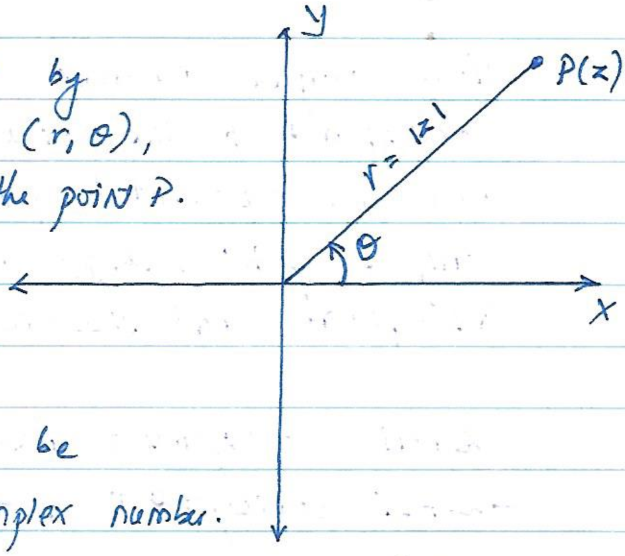
The x-axis and y-axis in the Argand plane are called, respectively, the real axis and the imaginary axis.

The representation of a complex number $z = x+iy$ and its conjugate $\bar{z} = x-iy$ in the Argand plane are $P(x,y)$ and $Q(x,-y)$

Geometrically, the point $(x,-y)$ is the mirror image of the point (x,y) on the real axis.



Polar representation of a complex number



The P is uniquely determined by ordered pair of real numbers (r, θ), called the polar coordinates of the point P.

$x = r\cos\theta \qquad y = r\sin\theta$

$\therefore z = x + iy = r(\cos\theta + i\sin\theta)$

$z = r(\cos\theta + i\sin\theta)$ is said to be the polar form of the complex number.

Here, $r = \sqrt{x^2+y^2} = |z|$ is the modulus of z and θ is called argument (or amplitude) of z. denoted by arg z.

The value of θ such that $-\pi < \theta \leq \pi$ is called principal argument of z and is denoted by arg z.

Quadratic Equations

Consider the quadratic equation $ax^2 + bx + c = 0$

let us assume $b^2 - 4ac < 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{4ac-b^2}\, i}{2a}$$

A polynomial equation has atleast one root.

"A ~~polynomial of this theorem~~

"A polynomial equation of degree n has n roots.

$\bar{z}$ - conjugate $= x - iy$

$|z|$ - modulus $= \sqrt{x^2+y^2}$

Example 1: Find the multiplicative inverse of $2-3i$

Sol^n: Let $z = 2-3i$ Inverse $z^{-1} = \frac{1}{z} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2}$

$$z^{-1} = \frac{2+3i}{(2)^2+(3)^2} = \frac{2+3i}{4+9} = \frac{2+3i}{13}$$

$$\therefore z^{-1} = \frac{2}{3} + \frac{3i}{13}$$

Example 2: Express the following in the form $a+ib$.

(i) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Sol^n: (i) $\frac{5+i\sqrt{2}}{1-i\sqrt{2}} = \frac{(5+i\sqrt{2})}{(1-i\sqrt{2})} \times \frac{(1+i\sqrt{2})}{(1+i\sqrt{2})}$ $[\because i^2 = -1]$

$$= \frac{5 + i5\sqrt{2} + i\sqrt{2} + i^2 2}{(1)^2 - (i\sqrt{2})^2}$$

$$= \frac{3+i6\sqrt{2}}{1+2} = \frac{\cancel{3}(1+i2\sqrt{2})}{\cancel{3}}$$

$$= 1 + i2\sqrt{2}$$

(ii) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{(i^2)^{17} \cdot i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \underline{i}$

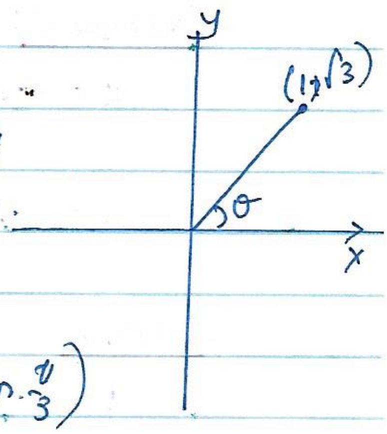
Example 3: Represent the complex number $z = 1+i\sqrt{3}$ in the polar form.

Sol^n: Let $1 = r\cos\theta$ $\sqrt{3} = r\sin\theta$

Squaring and adding, $r^2(\cos^2\theta + \sin^2\theta) = 4$

$r^2 = 4$ $r = \pm 2$

$\cos\theta = \frac{1}{2}$ $\sin\theta = \frac{\sqrt{3}}{2}$ $\theta = 60° = \frac{\pi}{3}$



$\therefore$ Required polar form $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Example 3: Convert the complex number $\frac{-16}{1+i\sqrt{3}}$ into polar.

Sol^n: $z = \frac{-16}{(1+i\sqrt{3})} \times \frac{(1-i\sqrt{3})}{(1-i\sqrt{3})} = \frac{-16(1-i\sqrt{3})}{1-(i\sqrt{3})^2}$

$= \frac{-16(1-i\sqrt{3})}{1+3} = \frac{-\cancel{16}^{4}(1-i\sqrt{3})}{\cancel{4}}$

$z = -4 + i4\sqrt{3}$

let $r\cos\theta = -4$ $\quad r\sin\theta = 4\sqrt{3}$

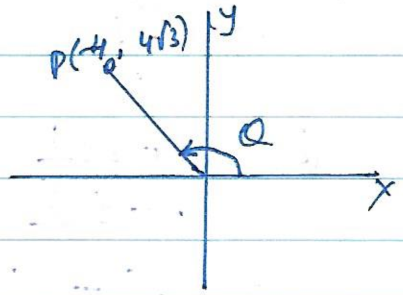
squaring and adding $\quad r^2(\cos^2\theta + \sin^2\theta) = (-4)^2 + (4\sqrt{3})^2$

$r^2 = 16 + 16\times 3$

$= 16 + 48$

$= 64$

$r = 8$



$$\begin{array}{r} \overset{1}{48} \\ 16 \\ \hline 64 \end{array}$$

Hence, $\cos\theta = \frac{-4}{8} = \frac{-1}{2}$ $\quad \sin\theta = \frac{4\sqrt{3}}{8\,2} = \frac{\sqrt{3}}{2}$

$\cos\theta = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$ $\quad$ Required polar form $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Example 4: Solve $x^2+2 = 0$

Sol^n: $x^2+2=0$, $\quad x^2 = -2$ $\quad x = \pm\sqrt{-2} = \pm i\sqrt{2}$

Example 5: Solve $x^2+x+1=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\quad b^2-4ac = (1)^2 - 4\cdot1\cdot1 = -3$

$x = \frac{-1\pm\sqrt{-3}}{2\times1} = \frac{-1\pm i\sqrt{3}}{2}$

Example 6: Solve $\sqrt{5}x^2+x+\sqrt{5}$. $\quad$ **Sol^n:** $b^2-4ac = 1 - 4\times\sqrt{5}\times\sqrt{5} = -19$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{-19}}{2\sqrt{5}} = \frac{-1\pm i\sqrt{19}}{2\sqrt{5}}$

Example 7: Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

Sol^n $z = \dfrac{(3-i2)(2+i3)}{(1+i2)(2-i)} = \dfrac{6+i9-i4+6}{2-i+i4+2}$

$$\begin{array}{r} 48 \\ 15 \\ \hline 63 \end{array}$$

$= \dfrac{12+i5}{4+i3}$

$= \dfrac{(12+i5)}{(4+i3)} \times \dfrac{(4-i3)}{(4-i3)} = \dfrac{48-36i+20i+15}{16-\cancel{12i}+\cancel{12i}+9}$

$= \dfrac{63-16i}{25} = \dfrac{63}{25} - \dfrac{16}{25}i$

$\therefore$ Conjugate of z is $\bar{z} = \dfrac{63}{25} + \dfrac{16}{25}i$

Example 8: Find the modulus and argument of the complex numbers:
(i) $\dfrac{1+2i}{1-i}$ (ii) $\dfrac{1}{1+i}$

Sol^n: (i) $\dfrac{1+i}{1-i} = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{1-1+2i}{1-i^2} = \dfrac{2i}{1+1}$

$= 0+i$

Modulus $= \sqrt{x^2+y^2} = \sqrt{(1)^2} = 1$

Let $r\cos\theta = 0$ $r\sin\theta = 1$

$r^2 = 1$ $r = 1$

$\cos\theta = 0$ $\sin\theta = 1$ $\theta = \dfrac{\pi}{2}$

$\therefore$ argument is $\dfrac{\pi}{2}$, modulus is 1

(ii) $\dfrac{1}{1+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1-i}{1+1} = \dfrac{1}{2} - i\dfrac{1}{2}$

Let $r\cos\theta = \dfrac{1}{2}$ $r\sin\theta = \dfrac{-1}{2}$ $r^2 = \dfrac{1}{4}+\dfrac{1}{4} = \dfrac{1}{2}$ $r_1 = \dfrac{1}{\sqrt{2}}$

$\therefore \cos\theta = \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $\sin\theta = \dfrac{-1}{\sqrt{2}}$ $\theta = \dfrac{-\pi}{4}$

Modulus is $\dfrac{1}{\sqrt{2}}$, argument is $\dfrac{-\pi}{4}$

Example 9: If $x+iy = \frac{a+ib}{a-ib}$ prove that $x^2+y^2=1$

Soln: $x+iy = \frac{a+ib}{a-ib} = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2}$

$$x+iy = \frac{a^2-b^2}{a^2+b^2} + i\frac{2ab}{a^2+b^2}$$

$$x-iy = \frac{a^2-b^2}{a^2+b^2} - i\frac{2ab}{a^2+b^2}$$

$$x^2+y^2 = (x+iy)(x-iy) = \left(\frac{a^2-b^2}{a^2+b^2} + i\frac{2ab}{a^2+b^2}\right)\left(\frac{a^2-b^2}{a^2+b^2} - i\frac{2ab}{a^2+b^2}\right)$$

$$= \left[\frac{a^2-b^2}{a^2+b^2}\right]^2 + \left(\frac{2ab}{a^2+b^2}\right)^2$$

$$= \frac{a^4+b^4-2a^2b^2+4a^2b^2}{(a^2+b^2)^2} = \frac{a^4+b^4+2a^2b^2}{(a^2+b^2)^2}$$

$$x^2+y^2 = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Example 10: Find real θ such that $\frac{3+i2\sin\theta}{1-i2\sin\theta}$ is purely real.

Soln: $\frac{3+2\sin\theta i}{1-2\sin\theta i} = \frac{3+2\sin\theta i}{(1-2\sin\theta i)} \times \frac{(1+2\sin\theta i)}{(1+2\sin\theta i)}$

$$= \frac{3+i6\sin\theta+i2\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$$

$$= \frac{3+i8\sin\theta-4\sin^2\theta}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + i\frac{8\sin\theta}{1+4\sin^2\theta}$$

Given complex number is real. Therefore, $\frac{8\sin\theta}{1+4\sin^2\theta} = 0$

$\sin\theta = 0 \quad \theta = n\pi, \; n \in Z.$

Example 11: Convert the complex number $z = \dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

Sol^n: we have $z = \dfrac{i-1}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \dfrac{-2+2i}{1+\sqrt{3}i}$

$$= \frac{2(-1+i)\times(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \frac{2(-1+\sqrt{3}i+i+\sqrt{3})}{1+3}$$

$$= \frac{(\sqrt{3}-1) + i(\sqrt{3}+1)}{2} = \frac{\sqrt{3}-1}{2} + i\frac{\sqrt{3}+1}{2}$$

put $r\cos\theta = \dfrac{\sqrt{3}-1}{2}$ $\quad r\sin\theta = \dfrac{\sqrt{3}+1}{2}$

Squaring and adding, $r^2 = \left(\dfrac{\sqrt{3}-1}{2}\right)^2 + \left(\dfrac{\sqrt{3}+1}{2}\right)^2$

$$r^2 = \frac{3+1-\cancel{2\sqrt{3}}+3+1+\cancel{2\sqrt{3}}}{4} = \frac{8}{4} = 2$$

$r^2 = 2$ which gives $r = \sqrt{2}$

$\cos\theta = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ $\quad \sin\theta = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Therefore $\theta = \dfrac{\pi}{4} + \dfrac{\pi}{6} = \dfrac{5\pi}{12}$ Hence, Polar form $\sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$

Example 12: If z_1, z_2 are complex numbers such that $\frac{2z_1}{3z_2}$ is purely imaginary number, then find $\left|\frac{z_1 - z_2}{z_1 + z_2}\right|$.

Solution: Since $\frac{2z_1}{3z_2}$ is purely imaginary.

$$\therefore \frac{2z_1}{3z_2} = ki \quad , k \in R.$$

$$\frac{z_1}{z_2} = \frac{3k}{2} i \quad \text{—— (1)}$$

Now $$\frac{z_1 - z_2}{z_1 + z_2} = \frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1} = \frac{\frac{3k}{2}i - 1}{\frac{3k}{2}i + 1} \quad \text{from (1)}$$

$$= \frac{-2 + 3ki}{2 + 3ki}$$

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \frac{|-2 + 3ki|}{|2 + 3ki|} = \frac{\sqrt{(-2)^2 + (3k)^2}}{\sqrt{(2)^2 + (3k)^2}} = \frac{\sqrt{4 + 9k^2}}{\sqrt{4 + 9k^2}}$$

$$= 1.$$

Example 13: If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that: $|w| = 1 \Rightarrow z$ is purely real.

Sol^n: $|w| = 1 \Rightarrow \left|\frac{1 - iz}{z - i}\right| = 1 \quad \therefore |1 - iz| = |z - i|$

$$\Rightarrow |1 - i(x + iy)| = |x + iy - i|$$

$$|(1 + y) - ix| = |x + i(y - 1)|$$

$$\sqrt{(1 + y)^2 + x^2} = \sqrt{x^2 + (y - 1)^2}$$

$$(1 + y)^2 + x^2 = x^2 + (y - 1)^2$$

$$\cancel{1} + \cancel{y^2} + 2y + \cancel{x^2} = \cancel{x^2} + \cancel{y^2} + \cancel{1} - 2y$$

$$4y = 0 \Rightarrow y = 0$$

Hence $z = x + iy = x + i0 = x$, which is purely real.