Chapter 6 : Applications of Derivatives (XII)

Calculus was created to describe how quantities change. Suppose that Q is a quantity that varies with time t, and write $Q = f(t)$, for the value of Q at time t.

For example, Q might be
(i) The volume of balloon being inflated
(ii) The amount of a chemical product produced in a reaction
(iii) The distance travelled in t hours after the beginning of a journey.

The change in Q from time t to time $t+\Delta t$ is the increment.
$$\Delta Q = f(t+\Delta t) - f(t)$$

The average rate of change of Q (per unit of time) is, by definition, the ratio of the change ΔQ in Q to the change Δt in t. Thus it is the quotient

$$\frac{\Delta Q}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

We define the instantaneous rate of change of Q (per unit of time) to be the limit of this average rate as $\Delta t \to 0$. That is, the instantaneous rate of change of Q is

$$\lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

But the right-hand limit in equation above is simply the derivative $f(t)$ $\therefore \frac{dQ}{dt} = f'(t)$

If two variables x and y vary with t, then by chain rule

$$\frac{dy}{dt} = \frac{dy}{dt} \div \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot \frac{dx}{dt}$$

Solved Examples

1) The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

Sol^n: Let x be the side of an equilateral triangle.

Given $\frac{dx}{dt} = 2$ cm/s

Area of equilateral triangle $A = \frac{\sqrt{3}}{4} x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \frac{dx}{dt} = \frac{\sqrt{3}}{2} x \frac{dx}{dt} = \frac{\sqrt{3}}{2} \times 20 \times 2 = 20\sqrt{3}$$

Hence, the rate of its area increasing $= 20\sqrt{3}$ cm^2/s

2) The radius of a spherical soap bubble is increasing at the rate of 0.3 cm/s. Find the rate of change of its: (i) Volume (ii) Surface-area.

Sol^n: Let r be the radius of the spherical soap bubble.

Given $\frac{dr}{dt} = 0.3$ cm/s

(i) Volume of sphere $V = \frac{4}{3} \pi r^3$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}} \pi \cancel{3} r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left(\frac{dV}{dt}\right)_{r=8} = 4\pi \, 8 \times 8 \times 0.3 = 76.8\pi \text{ cm}^3/\text{s}$$

(ii) Surface Area $S = 4\pi r^2$ $\quad \frac{dS}{dt} = 4\pi \, 2r \frac{dr}{dt}$

$$\left(\frac{dS}{dt}\right)_{r=8} = 4\pi \, 2 \times 8 \times 0.3 = 19.2\pi \text{ cm}^2/\text{s}$$

3) A particle moves along the curve $y = x^2+2x$. Find the point on the curve such that x and y co-ordinates of the particle change with the same rate.

Soln:

$$y = x^2+2x$$

$\frac{dy}{dt} = (2x+2)\frac{dx}{dt}$ Since $\frac{dy}{dt} = \frac{dx}{dt}$

$\therefore \quad 2x+2 = 1$

$2x = -1 \quad , \quad x = \frac{-1}{2}$

$y = \left(\frac{-1}{2}\right)^2 + 2\left(\frac{-1}{2}\right) = \frac{1}{4} - 1 = \frac{-3}{4}$

Hence, required point is $\left(\frac{-1}{2}, \frac{-3}{4}\right)$.

4) A particle moves along the curve $y = \frac{2}{3}x^3+1$.

Find the points on the curve at which y-coordinate is changing twice as fast as the x-coordinate.

Ans. $\left(1, \frac{5}{3}\right)$ and $\left(-1, \frac{1}{3}\right)$.

5) A stone is dropped into a quite lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of circular wave is 7.5 cm, how fast is the enclosed area increasing?

Soln: Let r be the radius of circular wave
A be the area. at any time t.

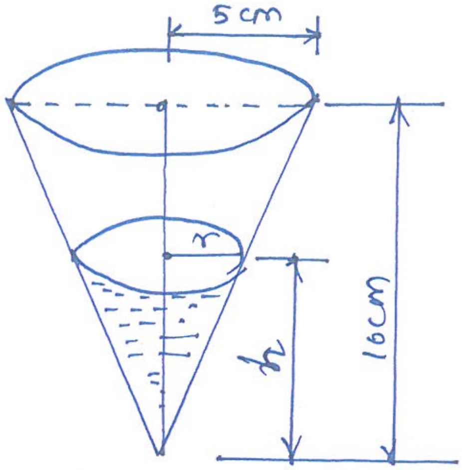
$A = \pi r^2$

$\frac{dr}{dt} = 3.5$ cm/s (given)

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \times 3.5 = 7\pi r$

$\left(\frac{dA}{dt}\right)_{r=7.5} = 7\times\pi\times 7.5 = 52.5\pi$ cm^2/sec.

(6) Water is leaking from a conical funnel at the rate of $5\ cm^3/sec$. If the radius of the base of the funnel is 5 cm and height 10 cm, find the rate at which the water level is dropping when it is 2.5 cm from the top.



Sol^n

Let at time t r be the radius of base and h be the height.

Volume of water $V = \frac{1}{3}\pi r^2 h$

Using the similar triangle properties,

$$\frac{h}{10} = \frac{r}{5} \Rightarrow r = \frac{1}{2}h$$

$$\therefore V = \frac{1}{3}\pi \frac{h^2}{4}.h = \frac{\pi}{12}h^3$$

$\therefore$ Volume of water at any time $V = \frac{\pi}{12}h^3$

$$\frac{dV}{dt} = \frac{\pi}{12}.3h^2\frac{dh}{dt} = \frac{\pi h^2}{4}.\frac{dh}{dt}$$

$\frac{dV}{dt} = -5\ cm^3/sec$ (Given)

$$\therefore \frac{\pi h^2}{4}\frac{dh}{dt} = -5 \Rightarrow \pi h^2\frac{dh}{dt} = -20$$

When water level is 2.5 cm from top, $h = 7.5$ cm.

$$\therefore \pi(7.5)^2\frac{dh}{dt} = -20 \Rightarrow \frac{dh}{dt} = -\frac{20}{\pi(7.5)^2} = \frac{-16}{45\pi}\ cm/sec$$

Hence, the rate at which the water is dropping is $\frac{16}{45\pi}$ cm/sec.

7) An airforce plane is ascending vertically at the rate of 100 km/hr. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 minutes after it started ascending?
(Visible area $A = \dfrac{2\pi r^2 h}{r+h}$, where h is the height of plane above the earth).

Soln: Since, the airforce plane is ascending vertically at the rate of 100 km/hr.

$\therefore$ Height of plane after 3 minutes $= \dfrac{100}{60} \times 3 = 5$ km.

Visible area $A = \dfrac{2\pi r^2 h}{r+h}$, where h is the height of plane above the earth. [r is fixed, h varies]

Now, $$\frac{dA}{dt} = 2\pi r^2 \left\{ \frac{(r+h) - h(1)}{(r+h)^2} \right\} \frac{dh}{dt} \text{ km}^2/\text{hr}.$$

$$= 2\pi r^2 \times \frac{r}{(r+h)^2} \times \frac{dh}{dt} = \frac{2\pi r^3}{(r+h)^2} \cdot \frac{dh}{dt} \text{ km}^2/\text{hr}.$$

$\dfrac{dh}{dt} = 100$ km/hr. after 3 minutes, when $h = 5$ km.

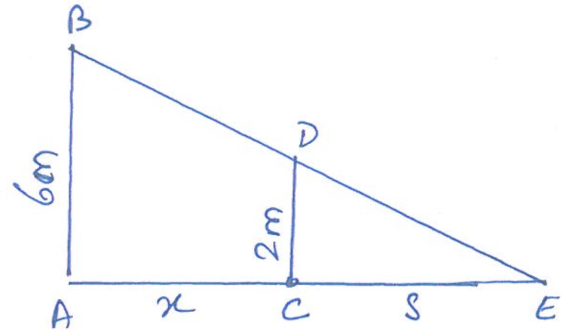
$$\frac{dA}{dt} = \frac{2\pi r^3}{(r+h)^3} \cdot 100 = \frac{200\pi r^3}{(r+h)^2} \text{ km}^2/\text{hr}.$$

$\therefore$ Rate of increase of visible area

$$\frac{dA}{dt} = \frac{200\pi r^3}{(r+5)^2} \text{ km}^2/\text{hr}.$$

8) A man of height 2 metres walks at a uniform speed of 5 km/hr away from a lamp-post which is 6 metres high. Find the rate at which the length of his shadow increases.

Sol^n: Let x be the distance of man from the pole at any time t.
Let s be the length of shadow.



Using the properties of similar triangle.

$$\frac{2}{6} = \frac{s}{x+s} \Rightarrow \frac{1}{3} = \frac{s}{x+s} \Rightarrow x+s = 3s$$

$$x = 2s \Rightarrow s = \frac{1}{2}x$$

Rate of increase of length of shadow

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} \quad \text{(chain rule).}$$

$$\frac{ds}{dx} = \frac{1}{2} \qquad \frac{dx}{dt} = 5 \text{ km/hr} = 5000 \text{ m/hr.}$$

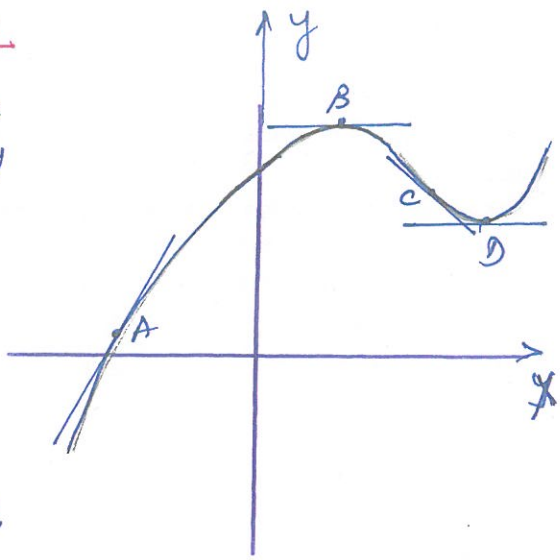
(Man is walking at the speed of 5 km/h).

$$\therefore \frac{ds}{dt} = \frac{1}{2} \times 5000 \text{ m/h}$$

$$= 2500 \text{ m/hr.}$$

$\therefore$ Rate of increase of length of shadow = 2.5 km/hr.

Increasing and Decreasing functions



The graph of $y = f(x)$ is shown in the figure and tangents are drawn at points A, B, C and D.

At points on the curve where the moving point is rising, we say that $y = f(x)$ is an increasing function. At these points, y increases as x increases.

At points where moving point is falling, $y = f(x)$ is a decreasing function.

At the point A, the function is increasing. Here the slope of tangent is positive.
At point C, the function is decreasing and the slope of the tangent is negative.
At B and D, the slope of the tangent is zero.
The points B and D separate rising and falling portions of the curve.

INCREASING FUNCTION

A function $f(x)$ is said to be an increasing function of x if:
$f(x)$ increases as x increases
or $f(x)$ decreases as x decreases.

Symbolically : $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$
$x_1 \ge x_2 \Rightarrow f(x_1) \ge f(x_2)$

If $f(x)$ is an increasing function, then x and $f(x)$ both increase or decrease simultaneously.
Example : $f(x) = x^2$ $x \ge 0$ is an increasing function of x because $f(x)$ increases as x increases.

$x =$	0	1	2	3	...
$f(x) =$	0	1	4	9	...

Theorem : for a function $f(x)$ continuous in $[a,b]$ if $f'(x) \geq 0$ for all x in (a,b), then $f(x)$ is an increasing function in $[a,b]$.

Strictly Increasing Function

A function $f(x)$ is said to be strictly increasing function of x if : $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

Theorem : For a function $f(x)$ continuous in $[a,b]$ if $f'(x) > 0$ for all x in (a,b), then $f(x)$ is strictly increasing function in (a,b).

Note : There can be functions, which are increasing but not strictly increasing.

DECREASING FUNCTIONS

A function $f(x)$ is said to be a decreasing function of x if : $f(x)$ increases as x decreases

or $f(x)$ decreases as x increases.

Symbolically : $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$ or

$x_1 \geq x_2 \Rightarrow f(x_1) \leq f(x_2)$

Example : $f(x) = \cos x$, $0 \leq x \leq \frac{\pi}{2}$ is a decreasing function because $f(x)$ decreases as x increases.

$x =$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f(x) =$	1	0.87	0.5	0

Theorem: For a function $f(x)$ continuous in $[a, b]$ if $f'(x) \leq 0$ for all x in (a, b), then $f(x)$ is a decreasing function in $[a, b]$.

Strictly Decreasing Function

A function $f(x)$ is said to be strictly decreasing function of x if: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

Theorem: For a function $f(x)$ continuous in $[a, b]$, if $f'(x) < 0$ for all x in (a, b) then $f(x)$ is strictly decreasing function in (a, b)

Note: There can be functions, which are decreasing but not strictly decreasing.

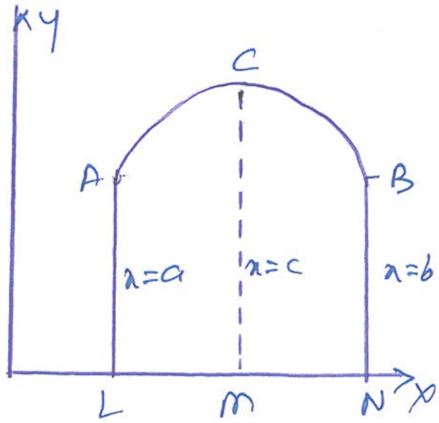
Remarks:

(i) If $f'(x) > 0$ for each $x \in (a, b)$, f is strictly increasing in (a, b)

(ii) If $f'(x) < 0$ for each $x \in (a, b)$, f is strictly decreasing in (a, b)

MIXED FUNCTION

A function may be increasing in a certain interval and decreasing in another interval. The function is neither wholly increasing nor wholly decreasing.



Such functions are called mixed function. $f(x)$ is increasing in $[a, c]$ and decreasing in $[c, b]$.

Monotone Function: A function $f(x)$ is said to be monotone if it is either increasing or decreasing.

Solved Examples

1) Without using the derivative, show that the function $f(x) = 7x - 3$ is a strictly increasing function in R.

Soln: Let x_1 and $x_2 \in R$.

Now $x_1 > x_2 \Rightarrow 7x_1 > 7x_2$

$\Rightarrow 7x_1 - 3 > 7x_2 - 3$

$\Rightarrow f(x_1) > f(x_2)$

Hence, 'f' is strictly increasing function in R.

2) The income I of a doctor is given by:

$$I = x^3 - 3x^2 + 5x.$$

Can an insurance agent ensure the growth of his income?

Soln: $I = x^3 - 3x^2 + 5x$

$$\frac{dI}{dx} = 3x^2 - 6x + 5$$

$$= 3x^2 - 6x + 3 + 2$$

$$= 3(x^2 - 2x + 1) + 2$$

$$\frac{dI}{dx} = 3(x-1)^2 + 2 > 0$$

Since $\frac{dI}{dx} > 0$ for all values of x

$\therefore$ the income I of the doctor is increasing for all values of x. Hence, insurance agent can ensure growth in the income of the doctor.

3) Determine for which values of 'x', the function $f(x) = x^3 - 24x + 7$ is strictly increasing or strictly decreasing.

Sol^n:

$$f(x) = x^3 - 24x + 7$$
$$f'(x) = 3x^2 - 24$$
$$= 3(x^2 - 8)$$

(i) for $f(x)$ to be strictly increasing function of x:

$f'(x) > 0$ i.e. $3(x^2-8) > 0$

$\Rightarrow x^2 > 8$

$\Rightarrow |x| > 2\sqrt{2}$

$\Rightarrow x < -2\sqrt{2}$ or $x > 2\sqrt{2}$

Hence, the function is increasing in the interval $(-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$

(ii) for $f(x)$ to be strictly decreasing function of x:

$f'(x) < 0$ i.e. $3(x^2-8) < 0$

$x^2 < 8$

$\Rightarrow |x| < 2\sqrt{2}$

$-x < 2\sqrt{2}$

$x > -2\sqrt{2}$ $\quad$ $x < 2\sqrt{2}$

$\Rightarrow -2\sqrt{2} < x < 2\sqrt{2}$

4) Find the intervals in which the function $f(x)$ is (i) strictly increasing
(ii) strictly decreasing

$$f(x) = x^3 - 12x^2 + 36x + 17$$

12
$\sim$
6×2

Sol^n:

$$f(x) = x^3 - 12x^2 + 36x + 17$$
$$f'(x) = 3x^2 - 24x + 36$$
$$= 3(x^2 - 8x + 12)$$
$$= 3(x^2 - 6x - 2x + 12)$$
$$f'(x) = 3(x-2)(x-6)$$

(i) For $f(x)$ to be strictly increasing function of x:

$$f'(x) > 0$$

$$\therefore \quad (x-2)(x-6) > 0$$
$$\Rightarrow \quad x < 2 \quad \text{or} \quad x > 6$$

Hence, $f(x)$ is increasing in the interval $(-\infty, 2) \cup (6, \infty)$

(ii) For $f(x)$ to be strictly decreasing function of x:

$$f'(x) < 0$$
$$\therefore \quad (x-2)(x-6) < 0$$
$$\Rightarrow \quad x > 2 \quad x < 6$$
$$\Rightarrow \quad 2 < x < 6$$

Hence, $f(x)$ is decreasing in the interval $(2, 6)$

5) Find the intervals in which the function :
$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is (a) strictly increasing (b) strictly decreasing.

Soln:
$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$f'(x) = 12x(x-2)(x+1)$$

when $x < -1$ $f'(x) = (-ve)(-ve)(-ve) = -ve$

Thus $f(x)$ is strictly decreasing.

When $x > -1$ $x < 0$

$-1 < x < 0$ $f'(x) = (-ve)(-ve)(+ve) = +ve$

$f(x)$ is strictly increasing.

When $x > 0$ $x < 2$

$0 < x < 2$ $f'(x) = (+ve)(-ve)(+ve) = -ve$

$f(x)$ is decreasing.

When $x > 2$ $f'(x) = (+ve)(+ve)(+ve) = (+ve)$

$f(x)$ is increasing.

$f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$

$f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$.

6) Find the intervals in which the function :
$f(x) = \sin^4 x + \cos^4 x \quad 0 \leq x \leq \frac{\pi}{2}$

is strictly increasing or decreasing.

<u>Sol^n</u> : $f(x) = \sin^4 x + \cos^4 x$

$f'(x) = 4\sin^3 x \cdot \cos x - 4\cos^3 x \cdot \sin x$

$= 4\sin x \cdot \cos x\,(\sin^2 x - \cos^2 x)$

$= 4\sin x \cdot \cos x\,(-\cos 2A)$ $[\because \cos 2A = \cos^2 A - \sin^2 A]$

$= 2\,(\sin 2A)\,(-\cos 2A)$ $[\because \sin 2A = 2\sin A \cdot \cos B]$

$= -2\sin 2A \cos 2A$

$f'(x) = -\sin 4x$

(i) $f(x)$ is strictly increasing if $f'(x) > 0$

$-\sin 4x > 0$

$\Rightarrow \sin 4x < 0$

$\Rightarrow \pi < 4x < 2\pi$

$\Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

sin x, cosec +ve	All (+ve)
tan x, cot x (+v)	cos x, sec x (+ve)

$f(x)$ is strictly increasing in $(\frac{\pi}{4}, \frac{\pi}{2})$

(ii) $f(x)$ is strictly decreasing function $f'(x) < 0$

$-\sin 4x < 0 \Rightarrow \sin 4x > 0$

$\Rightarrow 0 < 4x < \pi$

$0 < x < \frac{\pi}{4}$ $(0, \frac{\pi}{4})$

$f(x)$ is strictly decreasing in $(0, \frac{\pi}{4})$

7) Find the intervals in which :

$f(x) = \sin 3x - \cos 3x \qquad 0 < x < \pi$

is strictly increasing or decreasing.

Sol^n : $f(x) = \sin 3x - \cos 3x$

$f'(x) = \cos 3x\,(3) - (-\sin 3x)(3)$

$= 3\cos 3x + 3\sin 3x$

$f'(x) = 3(\cos 3x + \sin 3x)$

put $f'(x) = 0 \qquad 3(\cos 3x + \sin 3x) = 0$

$\cos 3x = -\sin 3x$

$\tan 3x = -1$

from class XI, we know that if $\tan x = \tan y$

then $x = n\pi + y$

$\therefore \tan 3x = -\tan \frac{\pi}{4}$

$= \tan\left(\pi - \frac{\pi}{4}\right)$

$\tan 3x = \tan \frac{3\pi}{4}$

$\therefore 3x = n\pi + \frac{3\pi}{4}$

Since $0 < x < \pi \quad \therefore 3x = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $\frac{11}{4}\pi \qquad [n = 0, 1, 2]$

$x = \frac{\pi}{4}, \frac{7\pi}{12}$ or $\frac{11}{12}\pi$

The points $x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$ divide the interval into four parts:

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$

in $\left(0, \frac{\pi}{4}\right)$ $f'(x) > 0$ $\Rightarrow$ $f(x)$ is strictly increasing

in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$ $f'(x) < 0$ $\Rightarrow$ $f(x)$ is strictly decreasing

in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ $f'(x) > 0$ $\Rightarrow$ $f(x)$ is increasing

in $\left(\frac{11\pi}{12}, \pi\right)$ $f'(x) < 0$ $\Rightarrow$ $f(x)$ is strictly decreasing.

Hence $f(x)$ is strictly increasing in

$$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$$

$f(x)$ is strictly decreasing in

$$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$$

8) Find the values of x for which $f(x) = x^x$; $x > 0$ is strictly increasing or decreasing.

Soln $f(x) = x^x$ which defined for $x > 0$

Domain of f $D_f = (0, \infty)$

$$f(x) = x^x = e^{\log x^x} = e^{x \log x}$$

$$f'(x) = e^{x \log x} \cdot \frac{d}{dx}(x \log_e x)$$

$$= e^{x \log x} \cdot \left[x \frac{1}{x} + \log_e x \,(1)\right]$$

$$= e^{x \log x}\left[1 + \log_e x\right]$$

$f'(x) = x^x(1+\log_e x)$

(i) for $f(x)$ is to be strictly increasing $f'(x)>0$

$\Rightarrow x^x(1+\log_e x) > 0$

$\Rightarrow 1+\log_e x > 0$ $\quad [\because x^x > 0$ for $x>0]$

$\log_e x > -1$

$x > e^{-1}$

$x > \frac{1}{e}$

Hence, $f(x)$ is strictly increasing on $\left(\frac{1}{e}, \infty\right)$.

(ii) for $f(x)$ is strictly decreasing $f'(x) < 0$

$\Rightarrow x^x(1+\log_e x) < 0$

$\Rightarrow 1+\log_e x < 0$ $\quad x^x > 0$ for $x > 0$

$\log_e x < -1$

$x < e^{-1}$

$x < \frac{1}{e}$

Hence $f(x)$ is strictly decreasing on $\left(0, \frac{1}{e}\right)$.

9) If a, b, c are real numbers, then find the intervals in which :

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

is strictly increasing or decreasing.

Sol^n: $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$

$$f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix}$$

$$+ \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$f'(x) = (x+b^2)(x+c^2) - b^2c^2 + (x+a^2)(x+c^2) - a^2c^2$$
$$+ (x+a^2)(x+b^2) - a^2b^2$$

$$f'(x) = 3x^2 + 2(a^2+b^2+c^2)x \, .$$

(i) for $f(x)$ is to be strictly increasing ... of x:

$$f'(x) > 0$$
$$\Rightarrow \quad 3x^2 + 2(a^2+b^2+c^2)x > 0$$
$$\Rightarrow \quad 3x\left(x + \frac{2}{3}(a^2+b^2+c^2)\right) > 0$$

$f'(x) = (+)(+ve) = (+ve)$ $\qquad$ $f'(x) = (-ve)(-ve) = +ve$

$\therefore \quad x > 0 \qquad x < -\frac{2}{3}(a^2+b^2+c^2)$

Hence $f(x)$ is strictly increasing in $\left(-\infty, -\frac{2}{3}(a^2+b^2+c^2)\right) \cup (0, \infty)$

(ii) for $f(x)$ to be strictly decreasing function of x :

$f'(x) < 0 \quad \Rightarrow \quad 3x^2 + 2x(a^2+b^2+c^2) < 0$

$$3x\left[x + \frac{2}{3}(a^2+b^2+c^2)\right] < 0$$

$$x\left[x + \frac{2}{3}(a^2+b^2+c^2)\right] < 0$$

$$\left(x - \left(-\frac{2}{3}(a^2+b^2+c^2)\right)\right)(x-0) < 0$$

$\Rightarrow \quad x < 0 \quad$ or $\quad x > -\frac{2}{3}(a^2+b^2+c^2)$

$\Rightarrow \quad -\frac{2}{3}(a^2+b^2+c^2) < x < 0$

Hence $f(x)$ is strictly decreasing in $\left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$

(10) Show that : $\frac{x}{1+x} < \log(1+x) < x$ for $x > 0$

Sol^n : Let $f(x) = \log(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \text{ for } x > 0$$

$\therefore$ $f'(x) > 0$ for all $x > 0$ and $f'(0) = 0$.

Thus $f(x)$ is increasing in $(0, \infty)$

Also $f(0) = 0$

$x > 0 \quad f(x) > f(0)$

$f(x) > 0$

$$\log(1+x) - \frac{x}{1+x} > 0 \Rightarrow \log(1+x) > \frac{x}{1+x} \quad — (1)$$

Again let $g(x) = x - \log(1+x)$

$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ for $x > 0$

$g'(x) > 0$ for all $x > 0$ and $g'(0) = 0$

Thus $g(x)$ is increasing in $(0, \infty)$.

Also $g(0) = 0$ $x > 0$ $g(x) > g(0) \Rightarrow g(x) > 0$.

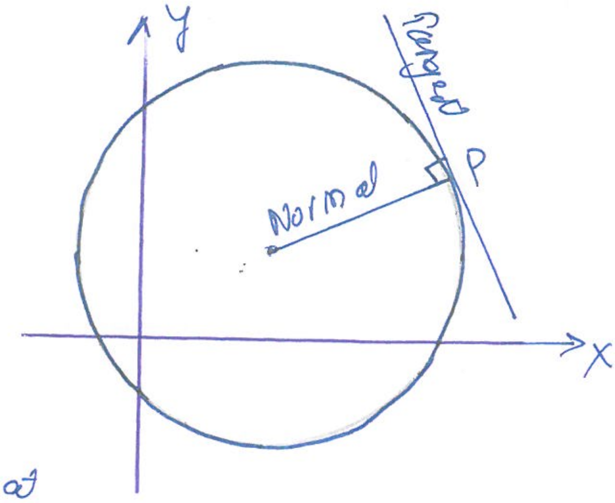
$\Rightarrow [x - \log(1+x)] > 0$

$x > \log(1+x)$ —— (2)

Combin (1) & (2) $\frac{x}{1+x} < \log(1+x) < x$ for $x > 0$

which is true.

Tangents and Normals



Now we shall use differentiation to find the equations of tangent and normal to a curve at a given point.

Let $y = f(x)$ be a curve.

Since $\frac{dy}{dx}$ = Slope of the tangent at any point,

$\therefore$ $\frac{dy}{dx}$ at $P = \tan\theta$, which is called as gradient

or $m = \tan\theta = \left(\frac{dy}{dx}\right)_P$

(1) (i) If a line makes an angle θ with positive x-axis, then its slope which is denoted by $m = \tan\theta$.

(ii) Slope of line which is parallel to x-axis is $\tan 0° = 0$ and the slope of a line parallel to y-axis is $\tan 90° = \infty$.

(iii) Slope of a line $ax+by+c=0$ is $-a/b$.

2. (i) Slope of a line joining two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$

(ii) Equation of line passing through (x_1, y_1) and having slope m is $y-y_1 = m(x-x_1)$.

3. If m_1 and m_2 are slopes of two lines, then acute angle θ between them is given by $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

4. Two lines having slopes m_1 & m_2 are parallel if $m_1 = m_2$ and are perpendicular if $m_1 m_2 = -1$.

Solved Examples

1) Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Ans:

$$y = 3x^4 - 4x$$

$$\text{Slope} = m = \frac{dy}{dx} = 12x^2 - 4 = 4(3x^2 - 1)$$

Slope of tangent at $x = 4$ is $\left(\frac{dy}{dx}\right)_{x=4} = 4(3 \times 16 - 1) = 764$.

2) Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Soln

$$\text{Slope of curve} = \frac{dy}{dx} = \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} \qquad \left(\frac{dy}{dx}\right)_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}.$$

3) Find the points on the curve $y = x^3 - 3x^2 + 2x$ at which tangent to the curve is parallel to the line $y - 2x + 3 = 0$.

Soln:

$$y = x^3 - 3x^2 + 2x$$

$$\frac{dy}{dx} = 3x^2 - 6x + 2$$

Slope of line $y - 2x + 3$ is $-b/a = 2$.

Since the lines are parallel $m_1 = m_2$

$$\therefore \quad 3x^2 - 6x + \cancel{2} = \cancel{2}$$

$$3x(x-2) = 0$$

$$x = 0, \quad x = 2$$

When $x = 0$, $y = 0$ $\quad x = 2$, $y = 8 - 12 + 4 = 0$

$\therefore$ Required points are $(0,0)$ $(2,0)$.

4) Find the slope of the normal to the curve $x = 1 - a\sin\theta$
$y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Sol^n: $x = 1 - a\sin\theta$ $\quad$ $y = b\cos^2\theta$

$\frac{dx}{d\theta} = 0 - a\cos\theta$ $\quad$ $\frac{dy}{d\theta} = b\,2(-\sin\theta)$

$= -a\cos\theta$ $\quad$ $= -2b\sin\theta\cdot\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-2b\sin\theta\cdot\cancel{\cos\theta}}{-a\cancel{\cos\theta}} = +\frac{b2}{a}\sin\theta$

$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}(1) = \frac{2b}{a}$

We know $m_1 m_2 = -1$ (When lines are perpendicular.

$\therefore \frac{2b}{a} m_2 = -1$

$m_2 = \frac{-a}{2b}$.

$\therefore$ Slope of normal $= \frac{-a}{2b}$.

5) Find the point on the parabola $y = (x-2)^2$ where the tangent is parallel to the chord joining $(2,0)$ & $(4,4)$.

Ans: $(3,1)$

6) Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Ans. $(2,-9)$.

7) Find the equation of all lines having slope 2 which is tangent to the curve $y = \frac{1}{x-3}$, $x \neq 3$.

[Ans. No line].

MISCELLANEOUS EXAMPLES

1) Show that the equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

2) Find the equation of the tangent to the curve $x^2 + 3y = 3$, which is parallel to the line $y = 4x + 5$.
[Ans. $4x - y + 13 = 0$].

3) Find the equation of the normal to the curve $y = \sin^2 x$ at the point $\left(\frac{\pi}{3}, \frac{3}{4}\right)$ [Ans. $24x + 12\sqrt{3}y - (8\pi + 9\sqrt{3}) = 0$.]

4) If the tangent to the curve $y = x^3 + ax + b$, at $P(1, -6)$ is parallel to the line $y - x = 5$, find the values of a and b.
[Ans $a = -2$ $b = -3$]

5) The equation of tangent at $(2, 3)$ on the curve $y^2 = ax^2 + b$ is $y = 4x - 5$. Find the value of a and b. [$a = 6$ $b = -15$]

6) Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distant from the origin. [Ans. a = constant].

7) At what points will the tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to the x-axis? Also, find the equations of tangents to the curve at these points.
[Ans. points $(2, 7)$ & $(3, 6)$ Eq. of tangents $y = 7$, $y = 6$.]

8) Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5$. [Ans. $48x - 24y = 23$].

9) Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) [Ans. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$]

10) Find the equations of the tangent and normal to the given curves $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$, $y_1 > 0$.
(24) [Ans. $9\sqrt{5}x - 24y + 14\sqrt{5} = 0$]

ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection of two curves at a point of intersection is the angle between the tangents of the two curves at that point.

If θ is the angle between two tangents, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where m_1, m_2 are values of $\frac{dy}{dx}$ for the two curves at the points $P(x_1, y_1)$.

Condition of tangency:

Two curves touch each other if the slopes of the tangents at the point of intersection are equal.

$\therefore$ The angle between the tangents i.e. $\theta = 0^\circ$

$\therefore \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 0^\circ = 0 \Rightarrow m_1 = m_2$

Condition of orthogonality:

Two curves are said to be orthogonal if they intersect at right angles.

The curve will be orthogonal if $\theta = \frac{\pi}{2}$

$\therefore \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 90^\circ = \infty \Rightarrow 1 + m_1 m_2 = 0$ i.e. $m_1 m_2 = -1$.

Rules for finding the angle of intersection between two curves

i) Find the point or points of intersection of the two curves by solving their equation simultaneously.

(ii) Differentiate the equations of the both the curve w.r.t x and dy/dx at the point or points of intersection and name them m_1 and m_2.

(iii) If θ is the angle between the curves, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(1) Find the angle of intersection of the following curves $xy = 6$ and $x^2y = 12$.

Soln: $xy = 6$ $\quad x^2 y = 12$

$y = \frac{6}{x}$ $\quad y = \frac{12}{x^2}$

Solving two equations $\frac{6}{\cancel{x}} = \frac{\cancel{12}^{2}}{x^{\cancel{2}}}$ $\quad x = 2$

$\therefore y = 3$

Given curves intersect at $(2, 3)$.

Differentiating $xy = 6$, $x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} = m_1$

Differentiating $x^2y = 12$ $\quad x^2\frac{dy}{dx} + y\,2x = 0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x} = m_2$

$m_1 = \left(\frac{dy}{dx}\right)_{(2,3)} = -\frac{3}{2}$ $\quad m_2 = \left(\frac{dy}{dx}\right)_{(2,3)} = -\frac{2\times 3}{2} = -3$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-3/2 - (-3)}{1 + -3/2 \times -3}\right| = \frac{3}{11} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{11}\right)$.

(2) Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Soln: Equation of curves $x = y^2$ —(1)

$xy = k$ —(2)

Solving (1) & (2)

$y^2 \times y = k \Rightarrow y = (k)^{1/3}$; $x = (k)^{2/3}$.

The point of intersection is $(k^{2/3}, k^{1/3}) = (x, y)$ (say)

Differentiating two curves (1) & (2)

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = m_1$$

$xy = k$ $$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} = m_2$$

As the curves cut at 90°

$$\therefore m_1 m_2 = -1$$

$$m_1 m_2 = \frac{1}{2\cancel{y}} \times \left(\frac{+\cancel{y}}{x}\right) = +1$$

$$\Rightarrow \frac{1}{2x} = +1$$

$$2x = 1 \Rightarrow 2(k)^{2/3} = 1$$

Cubing both sides:

$$\left[2(k)^{2/3}\right]^3 = 1^3$$

$$\Rightarrow 8k^2 = 1$$

3) Find the values of p for which curves $3x^2 = 9p(9-y)$ and $x^2 = p(y+1)$ cut each other at right angles.

[Ans. $p=0$, $p=4$]

4) Show that the equation of normal at any point t on curve $x = 3\cos t - \cos^3 t$

$y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$.

5) Find the equation of the normal at a point on the curve $x^2 = 4y$. which passes through the point $(1,2)$. Also find the equation of the corresponding tangent.

[Ans: $x+y=3$, $x-y=1$]

6) Find the equations of tangents to the curves $3x^2 - y^2 = 8$ which passes through the point $(\frac{4}{3}, 0)$

 [Ans: $3x-y=4$; $3x+y=4$]

Approximation by Differentials

Let $y = f(x)$

Then $y + \Delta y = f(x + \Delta x)$

$\Delta y = f(x + \Delta x) - y$

$\Delta y = f(x + \Delta x) - f(x)$

$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$

Setting $\Delta x = dx$, we get

$f(x + dx) = f(x) + \Delta y = f(x) + dy$ $\quad [\because \Delta y = dy]$.

$\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\,dx$

$\therefore f(x + dx) = f(x) + f'(x)\,dx$.

Exampel 1:

Using differentials find the approximate value of each of the following up to 3 places decimals.

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$.

Soln: (i) Take $y = \sqrt{x}$ $\quad x = 25$ $\quad \Delta x = 0.3$

$\Delta y = f(x + \Delta x) - f(x)$

$\therefore \Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{25.3} - \sqrt{25}$

$\Delta y = \sqrt{25.3} - 5$

$\Rightarrow \sqrt{25.3} = \Delta y + 5$

$\Delta y \approx dy$ $\quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{dx}{2\sqrt{x}} = \frac{0.3}{2\sqrt{25}} = \frac{0.3}{2 \times 5}$

$dy = 0.03$ $\quad \therefore \sqrt{25.3} = \underline{5.03}$

(ii) $\sqrt{49.5}$ Take $y = \sqrt{x}$

$x = 49 \quad \Delta x = 0.5$

$\Delta y = f(x + \Delta x) - f(x)$

$\Delta y = \sqrt{49.5} - \sqrt{49} = \sqrt{49.5} - 7$

$\therefore \sqrt{49.5} = 7 + \Delta y.$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{dx}{2\sqrt{x}} = \frac{0.5}{2\sqrt{49}} = \frac{0.5}{2 \times 7} = 0.036$

$\therefore \sqrt{49.5} = 7 + 0.036 = 7.036.$

(iii) $\sqrt{0.6}$ $y = \sqrt{x}$ $x = 0.64$ $\Delta x = -0.04.$

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{0.6} - \sqrt{0.64} = \sqrt{0.6} - 0.8$

$\therefore \sqrt{0.6} = 0.8 + \Delta y$

$dy = \frac{dx}{2\sqrt{x}} = \frac{-0.04}{2 \times \sqrt{0.64}} = -\frac{0.04}{2 \times 0.8} = \frac{-0.04}{1.6} = -0.025$

$\sqrt{0.6} = 0.8 + (-0.025) = 0.775.$

Example 2

Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(i) $(0.009)^{\frac{1}{3}}$ (ii) $(0.999)^{\frac{1}{10}}$ (iii) $(15)^{\frac{1}{4}}$

(i) $(0{\cdot}009)^{1/3}$ Let $y = x^{1/3}$ $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$

$x = 0{\cdot}008$ $\Delta x = 0{\cdot}001$

$\Delta y = (x+\Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}}$

$= (0{\cdot}009)^{\frac{1}{3}} - (0{\cdot}008)^{\frac{1}{3}}$

$= (0{\cdot}009)^{1/3} - 0{\cdot}2$

$(0{\cdot}009)^{\frac{1}{3}} = 0{\cdot}2 + \Delta y$

$dy = \frac{dx}{3x^{2/3}} = \frac{0{\cdot}001}{3\,(0{\cdot}008)^{2/3}} = \frac{0{\cdot}001}{3\times 0{\cdot}4} = \frac{0{\cdot}001}{0{\cdot}12}$

$dy = 0{\cdot}008$

$\therefore (0{\cdot}009)^{\frac{1}{3}} = 0{\cdot}2 + 0{\cdot}008 = 0{\cdot}208$ (approx).

(ii) $(0{\cdot}999)^{\frac{1}{10}}$ Let $y = x^{\frac{1}{10}}$

$x = 1$ $\Delta x = -0{\cdot}001$

$\Delta y = (x+\Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}} = (0{\cdot}999)^{\frac{1}{10}} - (1)^{\frac{1}{10}}$

$\therefore (0{\cdot}999)^{\frac{1}{10}} = 1 + \Delta y.$

$y = x^{\frac{1}{10}}$ $\frac{dy}{dx} = \frac{1}{10} \times x^{-\frac{9}{10}}$

$\therefore dy = \frac{-0{\cdot}001}{10} \times (1)^{-9/10} = -0{\cdot}0001$

$\therefore (0{\cdot}999)^{\frac{1}{10}} = 1 - 0{\cdot}0001 = 0{\cdot}9999.$

(iii) $(15)^{1/4}$ $\quad y = x^{1/4}$

$x = 16 \quad \Delta x = -1$

$\Delta y = (x+\Delta x)^{1/4} - x^{1/4} = (15)^{1/4} - (16)^{1/4} = (15)^{-\frac{1}{4}} - 2$

$(15)^{1/4} = 2 + \Delta y$

$y = x^{1/4} \quad \frac{dy}{dx} = \frac{1}{4} x^{-3/4} = \frac{1}{4x^{3/4}}$

$dy = \frac{-1}{4(16)^{3/4}} = \frac{-1}{4(2)^3} = \frac{-1}{32} = -0.03125$

$\therefore (15)^{-\frac{1}{4}} = 2 - 0.03125 = 1.969$ (approx.).

3) Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

$\begin{array}{r} 16 \\ 12 \\ \hline 28 \end{array}$

Soln: let $x = 2 \quad \Delta x = 0.01$

$\therefore \Delta y = f(x+\Delta x) - f(x)$

$\Delta y = f(2.01) - f(2) \Rightarrow f(2.01) = f(2) + \Delta y$.

$f(2) = 4(2)^2 + 5\times 2 + 2 = 4\times 4 + 10 + 2 = 16 + 12 = 28$

$\frac{dy}{dx} = 8x + 5 \Rightarrow dy = (8x+5)\,dx = (8\times 2+5)\,0.01$

$= (16+5)\times 0.01 = 0.21$

$\therefore f(2.01) = 28 + 0.21$

$= 28.21$.

4) Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 1%.

Soln:

Side of cube $= x$ m

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$dV = 3x^2\, dx$$

$dx = 1\%$ of $x = 0.01x$

$$\therefore dV = 3x^2 \times 0.01x$$

$$\Delta V = \underline{\underline{0.03x^3}} \text{ m}^3$$

5) Find the approximate change in the surface area of a cube of side x meters caused by decreasing the side by 1%.

Soln:

Side of cube $= x$ m

Decrease in side $\Delta x = 1\%$ of $x = -0.01x$

Surface area $S = 6x^2 \qquad \frac{dS}{dx} = 12x$

$$dS = 12x\, dx = 12x \times (-0.01x)$$

$$dS = -0.12x^2 \text{ m}^2.$$

6) If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

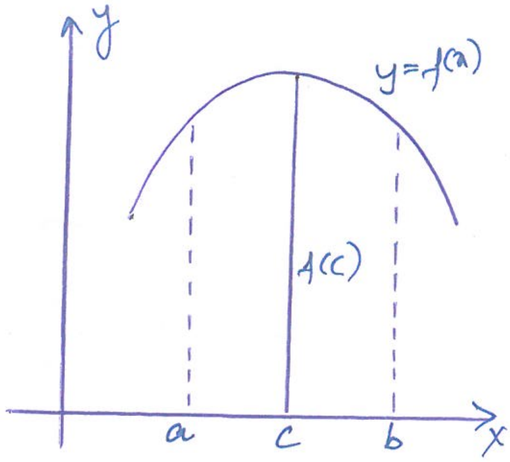
Soln: Let r be radius and Δr be error

$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = \frac{4}{3}\cdot 3\pi r^2 = 4\pi r^2 \qquad dV = 4\pi r^2\, dr$

$dV = 4\pi (7)^2 \times 0.02 = 3.92\pi \text{ m}^3 = 3.92 \times \frac{22}{7} = 12.32 \text{ m}^3$

$r = 7$ m $\qquad \Delta r = 0.02.$

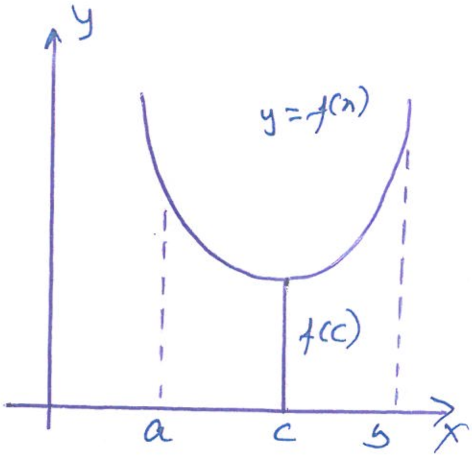
MAXIMA AND MINIMA



Let f be a function defined on an interval. Then,

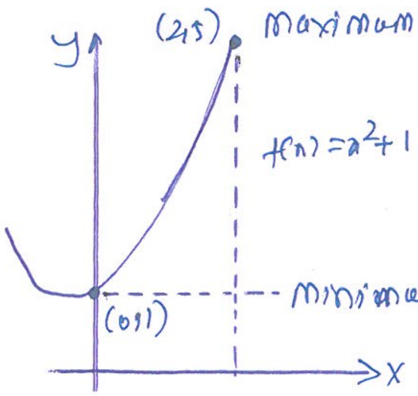
(i) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \geq f(x)$, $\forall\ x \in I$.

The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

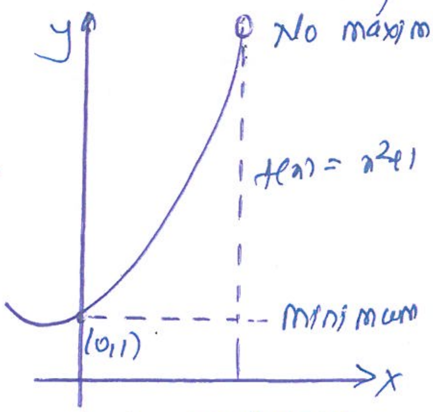


(ii) f is said to be a minimum value in I if there exists a point c in I such that $f(c) \leq f(x)$, $\forall\ x \in I$. The number $f(c)$ in this case is called the minimum value of f in I and point c is called a point of minimum value of f in I.

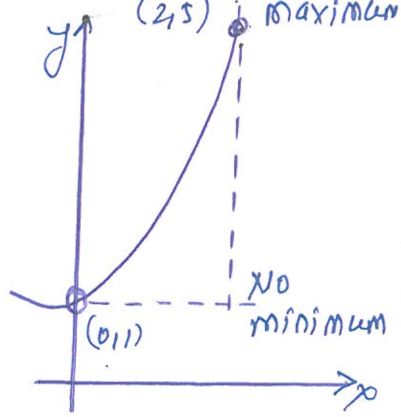
The <u>minimum</u> and <u>maximum</u> of a function on an interval, are the extreme values or extrema of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum on the interval, respectively.



f is continuous
$[-1, 2]$ is enclosed



f is continuous
$(-1, 2)$ is open



g is discontinuous
$[-1, 2]$ is closed.

Theorem : (The Extreme Value Theorem)

If f is continuous on a closed interval $[a,b]$, then f has both a minimum and maximum on interval.

Working rule to find absolute maximum or absolute minimum

Let f be a continuous function defined on an interval $I=[a,b]$ and let f be differentiable at all points of I, except possibly at the ends points a or b.
Then to find the maximum or minimum value of f

(i) Find all points, where $\frac{dy}{dx}=0$. Denote the solution by $x_1, x_2, ..$ where $x_1, x_2 \in I$

(ii) Take end points of the interval

(iii) All these points calculate the values of y.

(iv) Take the maximum and minimum values out of the values calculated in step (iii)

Example-1

Find the points of absolute maximum and minimum of:

$$y=(x-1)^{1/3}(x-2) \;;\; 1 \le x \le 9$$

Soln:

$$y=(x-1)^{1/3}(x-2)$$

$$\therefore \frac{dy}{dx}=(x-1)^{1/3}(1)+(x-2)\frac{1}{3}(x-1)^{-2/3}$$

$$=(x-1)^{-2/3}\left[(x-1)+\frac{1}{3}(x-2)\right]$$

$$=\frac{1}{3}(x-1)^{-2/3}(4x-5)$$

Now $\frac{dy}{dx} = 0$ gives $\frac{1}{3}(x-1)^{-2/3}(4x-5) = 0$

$\Rightarrow 4x-5 = 0$; $x = \frac{5}{4} \in [1, 9]$

$y_{at\ x=1} = 0$

$y_{at\ x=\frac{5}{3}} = \left(\frac{5}{4}-1\right)^{1/3}\left(\frac{5}{4}-2\right) = \left(\frac{1}{4}\right)^{\frac{1}{3}}\left(-\frac{3}{4}\right) = \frac{-3}{4\sqrt[3]{4}}.$

$y_{at\ x=9} = (9-1)^{1/3}(9-2) = (8)^{1/3}(7) = (2)(7) = 14$

Hence, 'y' is absolute maximum at $x=9$ and absolute minimum at $x = \frac{5}{4}$.

Example-2

Determine the absolute maximum and absolute minimum values of each of the following :

$$y = \frac{1}{2}x^2 + 5x + \frac{3}{2} \qquad -6 \le x \le -2.$$

Sol^n: $y = \frac{1}{2}x^2 + 5x + \frac{3}{2}$; $\frac{dy}{dx} = \frac{1}{2}\cdot 2x + 5 = x+5$

$\frac{dy}{dx} = 0$ which gives $x+5=0$, $x = -5 \in [-6, -2]$

$y_{-6} = \frac{1}{2}(-6)^2 + 5(-6) + \frac{3}{2} = -\frac{21}{2}$

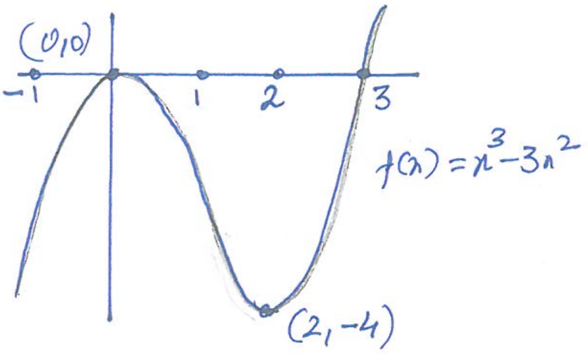
$y_{-5} = \frac{1}{2}(-5)^2 + 5(-5) + \frac{3}{2} = -11$

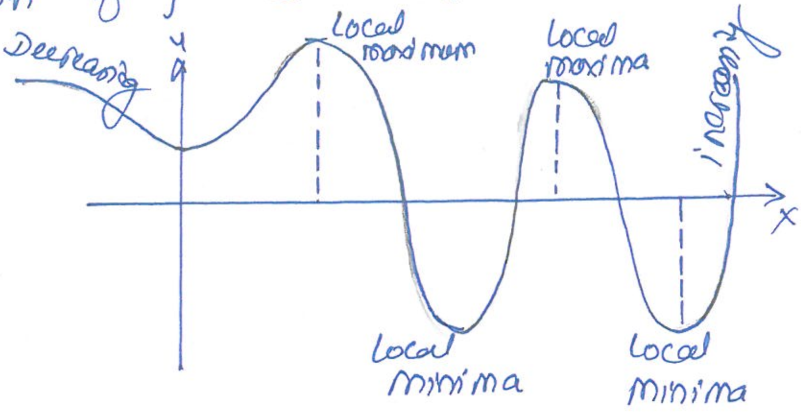
$y_{-2} = \frac{1}{2}(-2)^2 + 5(-2) + \frac{3}{2} = -\frac{13}{2}$

Hence, the absolute maximum value is $y = -\frac{13}{2}$ and absolute minimum value of $y = -11$.

LOCAL (RELATIVE) MAXIMA AND MINIMA

Roughly speaking, $f(c)$ is a local (relative) maximum value of f at c if the graph of f has a little hill above the point c. Similarly, $f(c)$ is a local minimum value of f at c if the graph of f has a little value above the point c.





More precisely, let f be real valued and let c be an interior point in the domain of f. Then :

(a) c_0 is called a point of local maxima if there is a $h>0$ such that $f(c_0) \geq f(x)$ for all x in (c_0-h, c_0+h). The value of $f(c_0)$ is called the local maximum value of f.

(b) c_0 is called a point of local mimima if there is a $h>0$ such that $f(c_0) \leq f(x)$, for all x in (c_0-h, c_0+h). The value of $f(c_0)$ is called the local minimum value of f.

(c) A point D_f, which is either a point of local maxima or local minima, is called a <u>extreme point</u> and the value of the function at this point is called an <u>extreme value</u>.

The graph of a function has a peak (or trough) at a point according as the point is of local maxima or local minima.

(d) Critical point : A point $c \in D_f$ is said to be critical (or turning or saddle) point of a continuous function 'f' if either $f'(c)=0$, $f'(c)$ is infinite or $f'(c)$ does not exist.

Stationary values are the values of $f(x)$ for those points at which $f'(x) = 0$.

Finding the Points of Local Maxima or Local Minima

(a) First Derivative Test

(i) Let function be $y = f(x)$. Find $\frac{dy}{dx}$.
Put $\frac{dy}{dx} = 0$. Solve it for x giving $x = a, b, c \ldots$

(ii) Select $x = a$ (say)
Study the sign of $\frac{dy}{dx}$ when (i) $x < a$ slightly
(ii) $x > a$ slightly.

(a) If the former is +ve and latter is −ve, then $f(x)$ is max. at $x = a$.
(b) If the former is −ve and latter is +ve, then $f(x)$ is min. at $x = a$.

(iii) Putting these values of 'x' for which $f(x)$ is max or min and get the corresponding max or min value of $f(x)$.

Example

Find the points of local maxima and local minima, if any, of the function: $f(x) = (x-1)(x+2)^2$.
Find also the local maximum and local minimum values.

Sol^n:
$$f(x) = (x-1)(x+2)^2$$
$$f'(x) = (x-1)(2)(x+2) + (x+2)^2(1)$$
$$= (x+2)[2x-\cancel{2} + x+\cancel{2}]$$
$$f'(x) = 3x(x+2)$$
$$f'(x) = 0 \qquad 3x(x+2) = 0 \qquad \Rightarrow \qquad x = 0, -2.$$

$f'(x) = 3x(x+2)$

When x is slightly less than 0, then $f'(x)$ is $-ve$.
When x is slightly greater than 0, then $f'(x)$ is $+ve$.

Thus $f'(x)$ changes from $-ve$ to $+ve$.

So $x = 0$ is a point of local minima and local minimum value $= f(0) = (0-1)(0+2)^2 = 4$.

at $x = -2$,
When x is slightly less than -2 $f'(x)$ is $+ve$.
When x is slightly greater than -2, $f'(x)$ is $-ve$.

Thus $f'(x)$ changes sign from $+ve$ to $-ve$.
So $x = -2$ is a point of local maxima and local maximum value is $f(-2) = (-2-1)(-2+2)^0 = 0$.

(b) **SECOND DERIVATIVE TEST**

(i) Put $y =$ given function $f(x)$ and find $\frac{dy}{dx}$ i.e. $f'(x)$

(ii) Put $\frac{dy}{dx} = 0$ i.e. $f'(x) = 0$ and solve for x giving $x = a, b, c, \ldots$

(iii) Select $x = a$. Find $\frac{d^2y}{dx^2}$ i.e. $f''(x)$ at $x = a$.

(a) if $\left(\frac{d^2y}{dx^2}\right)_{x=a}$ i.e. $f''(a)$ is $-ve$, $x = a$ gives the maximum value of the function.

(b) if $\left(\frac{d^2y}{dx^2}\right)_{x=a}$ i.e. $f''(a)$ is $+ve$, $x = a$ gives the minimum value of the function.

Similarly for $a = b, c, \ldots\ldots$

Example

Find all the points of local maxima and local minima of the function 'y' given by:

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

Sol^n :

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

$$f'(x) = 6x^2 - 12x + 6$$

$$f'(x) = 6(x-1)^2$$

$$f''(x) = 12(x-1)$$

$f'(x) = 0$ gives $x = 1$.

$$f''(1) = 0$$

Thus $x=1$ is neither a point of maxima nor of minima.

Now $f'''(x) = 12$

$$f'''(x)\Big]_{x=1} = 12 \neq 0$$

Hence, $x=1$ is a point of inflexion.

Solved Examples

1) Find the maximum and minimum values if any of the following functions without using derivatives:

(i) $f(x) = (2x-1)^2+3$

(ii) $f(x) = 16x^2-16x+28$

(iii) $f(x) = -|x+1|+3$

(iv) $f(x) = \sin 2x+5$

(v) $f(x) = \sin(\sin x)$

Soln: (i) $f(x) = (2x-1)^2+3$ $\quad D_f = R$.

Now $f(x) \geq 3$ $\quad [\because (2x-1)^2 \geq$ for all $x \in R]$

Hence, the minimum value $= 3$.
However, maximum value does not exist
since $f(x)$ can be made as large as we please.

(ii) $f(x) = 16x^2-16x+28$ $\quad D_f = R$
$= 16\left(x-\frac{1}{2}\right)^2+24$

$f(x) \geq 24$ $\quad \because 16\left(x-\frac{1}{2}\right)^2 \geq 0$ for all $x \in R$.

Hence, the minimum value is 24.

However, maximum value does not exist.

(iii) $f(x) = -|x+1| + 3$

$\Rightarrow f(x) \leq 3$ since $-|x+1| \leq 0$.

Hence, the maximum value $= 3$.

However, the minimum value does not exist.

(iv) $f(x) = \sin 2x + 5$.

Since $-1 \leq \sin 2x \leq 1$ for $x \in R$.

$-1+5 \leq \sin 2x + 5 \leq 1+5$

$4 \leq \sin 2x + 5 \leq 6$

$4 \leq f(x) \leq 6$

Hence Maximum value is $= 6$

minimum value $= 4$.

(v) $f(x) = \sin(\sin x)$

We know that $-1 \leq \sin x \leq 1$

$\sin(-1) \leq \sin(\sin x) \leq \sin(1)$

$\therefore -\sin(1) \leq f(x) \leq \sin(1)$

max value $= \sin(1)$

min value $= -\sin(1)$.

2) Find the maximum and minimum values if any, of the following functions by:

(i) $f(x) = (2x-1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ (iv) $f(x) = x^3 + 1$

Sol^n: (i) $f(x) = (2x-1)^2 + 3$

$$(2x-1)^2 \geq 0 \quad \forall\ x \in R.$$

$$(2x-1)^2 + 3 \geq 0 + 3$$

$$f(x) \geq 3$$

The minimum value of $f(x)$ is 3 and is obtained when $2x-1 = 0$ i.e. $x = \frac{1}{2}$.

$f(x)$ has no maximum value since $f(x) = (2x-1)^2 + 3 \to \infty$ as $x \to \infty$ and therefore, does not exist.

(ii) $f(x) = 9x^2 + 12x + 2 = [(3x)^2 + 2 \cdot 3x \cdot 2 + (2)^2] - 2$

$= (3x+2)^2 - 2$

Minimum value of $f(x)$ is -2 and is obtained when $3x+2 = 0$ i.e. $x = \frac{-2}{3}$

There is no maximum value of x because $f(x) \to \infty$ as $|x| \to \infty$.

(iii) $f(x) = -(x-1)^2 + 10$

$f(x) = 10 - (x-1)^2$

$(x-1)^2 \geq 0$

$-(x-1)^2 \leq 0$

$10 - (x-1)^2 \leq 0 + 10$

$f(x) \leq 10$.

Maximum value of $f(x)$ is 10 and is obtained when $x-1 = 0$ $x = 1$.

$f(x)$ has no minimum value.

(iv) $f(x) = x^3 + 1$ $\quad D_f = R$.

Let $x_1, x_2 \in R$ with $x_1 < x_2$

$x_1 < x_2 \Rightarrow x_1^3 < x_2^3 \Rightarrow x_1^3 + 1 < x_2^3 + 1 \Rightarrow f(x_1) < f(x_2)$

As $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ so f is increasing throughout its domain.

Therefore, f has neither a maximum nor a minimum value.

3) Prove that the following functions do not have maxima or minima :

(i) $f(x) = e^x$ $\quad$ (ii) $g(x) = \log x$ $\quad$ (iii) $h(x) = x^3 + x^2 + x + 1$

(iv) $f(x) = x + 2$.

Soln: (i) $f(x) = e^x$ $\quad f'(x) = e^x$

Now $f'(x) = 0$ $\quad e^x = 0$

But there is no real value of x for which $e^x = 0$ since $e^x > 0 \ \forall x \in R$.

So, there is no critical points, and therefore, $f(x)$ does not have maxima or minima.

(ii) $g(x) = \log x$ $\quad g'(x) = \frac{1}{x}$

For maxima or minima. $g'(x) = 0$ $\quad \frac{1}{x} = 0$

$\frac{1}{x} \neq 0$ for any $x \in R$.

So there is no critical point and hence $f(x)$ has no maxima or minima.

(iii) $h(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1$

$h'(x) = 0 \quad \Rightarrow \quad 3x^2 + 2x + 1 = 0$

$$x = \frac{-2 \pm \sqrt{4 - 4 \cdot 3 \cdot 1}}{6} = \frac{-2 \pm \sqrt{-8}}{6}$$

$x = \frac{-2 \pm 2\sqrt{2}\, i}{6} = \frac{-1 + \sqrt{2}\, i}{3}$ which is imaginary.

Thus there is no real value of x which satisfies $f'(x) = 0$

So there is no critical point and therefore, $f(x)$ does not have maxima or minima.

(iv) $f(x) = x + 2$ $\quad f'(x) = 1 \neq 0$ for any real value of x.

There is no critical point, therefore $f(x)$ does not have maxima or minima.

4) Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.

(i) $f(x) = x^3,\ x \in [-2, 2]$ $\quad$ (ii) $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$ $\quad$ (iv) $f(x) = (x-1)^2 + 3,\ x \in [-3, 1]$

Soln: (i) $f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad f'(x) = 0$

$x = 0 \in [-2, 2]$

$f(-2) = -8$ $\quad f(0) = 0$ $\quad f(2) = 8$

$\therefore$ absolute maximum value of $f(x) = 8$ at $x = 2$ and absolute minimum value of $f(x) = -8$ at $x = -2$.

(ii) $f(x) = \sin x + \cos x \quad x \in [0, \pi]$

$f'(x) = \cos x - \sin x \qquad f'(x) = 0 \qquad \cos x - \sin x = 0$

$\tan x = 1$

$\therefore x = \frac{\pi}{4} \in [0, \pi]$

$f(0) = \sin 0 + \cos 0$

$= 0 + 1$

$f(0) = 1$

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

$f(\pi) = \sin\pi + \cos\pi$

$= 0 + (-1)$

$= -1$

Absolute maxima $f(x) = \sqrt{2}$ at $x = \frac{\pi}{4}$

Absolute minima $f(x) = -1$ at $x = \pi$.

(iii) $f(x) = 4x - \frac{1}{2}x^2 \quad x \in [-2, \frac{9}{2}]$

$f'(x) = 4 - \frac{1}{2}\cdot 2x = 4 - x \qquad f'(x) = 0 \qquad 4 - x = 0$

$x = 4 \quad [-2, \frac{9}{2}]$

$f(-2) = 4(-2) - \frac{1}{2}(-2)^2 = -8 - 2 = -10$

$f(4) = 4\times 4 - \frac{1}{2}\,4\times 4 = 16 - 8 = 8$

$f(\frac{9}{2}) = 4\times\frac{9}{2} - \frac{1}{2}\times\frac{9}{2}\times\frac{9}{2} = \frac{9}{2}\left(4 - \frac{9}{4}\right) = \frac{9}{2}\left(\frac{16-9}{4}\right) = \frac{9}{2}\times\frac{7}{8}$

$= \frac{63}{8}$

Maximum value is 8 at $x = 4$

Minimum value is -10 at $x = -2$.

(iv) $f(x) = (x-1)^2 + 3 \quad x \in [-3, 1]$

$f'(x) = 2(x-1) \qquad f'(x) = 0 \qquad x = 1 \in [-3, 1]$

$f(-3) = 19 \qquad f(1) = 3$

Maximum value $= 19$ at $x = -3$

Minimum value $= 3$ at $x = 1$.

5) Find the maximum and minimum value of
$f(x) = x^3 - 12x^2 + 36x + 17$ in $1 \le x \le 10$.

[Ans. maximum = 177 ; minimum = 17].

6) Determine the maximum and minimum value of:

(i) $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ $0 \le x \le 2$ (ii) $y = 2\cos 2x - \cos 4x$ $0 \le x \le \pi$

(Ans. maximum = $\sqrt{2}$
minimum = 1) (maximum = $\frac{3}{2}$
minimum = -3)

7) Find the points of maximum and minimum of each of the following:

(i) $y = \sqrt{5}(\sin x + \cos 2x)$ $0 \le x \le \frac{\pi}{2}$ (maximum at $x = \sin^{-1}\frac{1}{4}$
minimum at $x = \frac{\pi}{2}$)

(ii) $y = (x-1)^{-\frac{1}{3}}(x-2)$ $1 \le x \le 9$. (maxi at $x = 9$ min at $x = \frac{5}{4}$).

8) Find the local extremum value of the following:

(i) The constant function α (ii) $\dfrac{1}{x^2+2}$.

(iii) x^2 (iv) $(x-1)(x+2)^2$ (v) $x^3 - 6x^2 + 9x + 15$

(vi) $(x-3)^4$ (vii) $x^3(2x-1)^3$ (viii) $\frac{x}{2} + \frac{2}{x}$, $x > 0$

Soln: (i) $f(x) = \alpha$ $f'(x) = 0$ for all x.

Let c be any real number, then $f'(c) = 0$
When x is slightly less than c $f'(c) = 0$
When x is slightly $> c$ $f'(c) = 0$
$\therefore$ c is neither a point of local maximum nor a point of local minimum. $f(x)$ has neither local maxima nor local minima

(ii) $f(x) = \frac{1}{x^2+2} = (x^2+2)^{-1}$

$f'(x) = -(x^2+2)^{-2} \cdot 2x = \frac{-2x}{(x^2+2)^2}$

for maxima or minima, $f'(x) = 0$

$\therefore \frac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$.

When x is slightly < 0 $f'(x) > 0$ +ve
when x is slightly > 0 $f'(x) < 0$ −ve

$\therefore$ $f'(x)$ changes sign from positive to negative as x increases through 0.

$\therefore$ $x = 0$ is a point of local maximum.
Local maximum value $f(0) = \frac{1}{2}$.

(iii) $f(x) = x^2$ $f'(x) = 2x$ $f'(x) = 0 \Rightarrow x = 0$

When x is slightly < 0, let $x = -0.1$
$f'(-0.1) = 2(-0.1) = -0.2$ which −ve

When x is slightly > 0, let $x = 0.1$
$f'(0.1) = 2(0.1) = 0.2$ +ve

$f'(x)$ changes sign from −ve to +ve as x increases through 0.
Hence, $x = 0$ is a point of local minima
local minimum value $= f(0) = 0$.

(iv) Ans: $x = 0$ is a point of local minima
local minimum value is -4.

$x = -2$ is a point of local maximum
local maximum value is 0.

(v) Ans : $x = 1, 3$

$x = 1$ is a point of local maxima
local maximum value $= 19$.

$x = 3$ is a point of local minima
local minima value is 15.

(vi) $x = 3$, point of local minima
local minimum value is 0.

(vii) $x = 0, \frac{1}{2}, \frac{1}{4}$

$x = 0$ is a point of inflexion
since $f'(x)$ does not change sign.

$x = \frac{1}{2}$ is a point of inflexion. $f'(x)$ does not change sign.

$x = \frac{1}{4}$ is a point of local minima
local minimum value is $\frac{1}{512}$

(viii) $x = -2, 2$
$x = 2$ is a point of local minima
local minimum value is 2.

9) Find the local maximum and minimum
(i) $y = x^5 - 5x^4 + 5x^3 - 1$ (Use second derivative method)

(ii) $y = (1-x)^2 e^x$ (Use second derivative method).

Ans : (i) $x = 1$ is point of local maxima and value is 0
$x = 3$ is the point of local minima and value is -28.

Applied Minimum and Maximum Problems

1) Find the two positive numbers whose sum is 16 and sum of whose cubes is minimum.

Sol^n: Let one number be x

then other number is $16-x$

Since numbers are positive $0 < x < 16$.

Sum of cubes $S = x^3 + (16-x)^3$

$$\frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1)$$

$$= 3x^2 - 3(16-x)^2$$

For max or min $\frac{dS}{dx} = 0$

$$\therefore \quad 3x^2 - 3(16-x)^2 = 0$$

$$x^2 = (16-x)^2$$

$$x = 16 - x$$

$$2x = 16$$

$$x = 8$$

$$\frac{d^2S}{dx^2} = 3 \cdot 2x - 3(2)(16-x)(-1)$$

$$= 6x + 6(16-x)$$

$$\left(\frac{d^2S}{dx^2}\right)_{x=8} = 6 \times 8 + 6(16-8) = 48 + 6 \times 8 = 96 > 0 \text{ +ve}$$

$\therefore$ $x = 8$ gives minimum value

$\therefore$ S is minimum when $x = 8$

other number is $16 - 8 = 8$.

2) Find two positive numbers x and y such that their sum is 35 and product x^2y^5 is maximum.

Soln: $x+y=35 \quad x>0 \quad y>0$

$P = x^2y^5 = x^2(35-x)^5$

$\frac{dP}{dx} = x^2\, 5(35-x)^4(-1) + (35-x)^5 \cdot 2x$

$= 2x(35-x)^5 - 5x^2(35-x)^4$

$\frac{dP}{dx} = 0$ for max or min

$\therefore \quad 2\cancel{x}(35-x)^5 - 5x^{\cancel{2}}(35-x)^4 = 0$

$x(35-x)^4\,[2(35-x) - 5x] = 0$

$x(35-x)^4(70-7x) = 0 \Rightarrow x = 0, 35, 10.$

$x \neq 0$ since $x>0$.

$x \neq 35$ since $y = 35-35 = 0$ but $y>0$

When x is slightly less than 10

$\frac{dP}{dx} = (+ve)(+ve)(+ve) = +ve$

When x is slightly >10 $\frac{dP}{dx} = (+ve)(+ve)(-ve) = -ve$

Thus $\frac{dP}{dx}$ change sign from $(+ve)$ to $(-ve)$ as x increases through 10. $\therefore$ P is maximum when $x=10$.

When $x=10$ $y=25$.

3) (i) Find two numbers whose sum is 24 and whose product is as large as possible.

(ii) Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

Ans (i) $x=y=12$

(ii) $x=15$, $y=45$.

4) Divide a number 15 into two parts that square of one multiplied with the cube of the other is maximum.

[Hint.. $y=x^2(15-x)^3$ one part $=6$, other part $=9$).

5) Divide 64 into two parts such that the sum of the cubes of two parts is minimum.

[Hint. $z=x^3+(64-x)^3$ one part $=32$ other part $=32$].

6) A projectile is fired upwards. Its height above the surface of the earth at time t is given by $h(t)=at^2+bt+c$ where a,b,c are non-zero constants and $a<0$. Determine how high will the projectile travel.

Soln:

$$h(t)=at^2+bt+c$$
$$h'(t)=2at+b$$

$h'(t)=0$ $\quad 2at+b=0$

$$t=-\frac{b}{2a}$$

$h''(t)=2a=-ve$ $[\because a<0]$

$\therefore$ Height is maximum at $t=-\frac{b}{2a}$

$$\left(h(t)\right)_{t=-\frac{b}{2a}}=a\left(\frac{-b}{2a}\right)^2+b\left(\frac{-b}{2a}\right)+c$$

$$=\frac{4ac-b^2}{4a}.$$

$\therefore$ Projectile will travel to a height of $\frac{4ac-b^2}{4a}$.

7) The rate of a chemical reaction y is given by the formula $y = kx(a-x)$ where x is the amount of product, a is the amount of material at the beginning of the reaction and $k > 0$. Determine the value of x for which the rate of chemical reaction will be maximum.

Soln : $y = kx(a-x)$

$$\frac{dy}{dx} = k[x(-1) + (a-x)(1)]$$

$$= k[-x + a - x]$$

$$= k[a-2x]$$

for max or min $\frac{dy}{dx} = 0 \Rightarrow k(a-2x) = 0$

$$x = \frac{a}{2}$$

$$\frac{d^2y}{dx^2} = k[-2] = -2k < 0$$

$\therefore$ y is maximum at $x = \frac{a}{2}$.

Hence, the rate of chemical reaction is maximum at $x = \frac{a}{2}$

8) The combined resistance R of two resistors is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ where R_1 and R_2 are the respective resistance of the two resistors with $R_1 + R_2 = a$ ($R_1, R_2 > 0$), where 'a' is a constant. Show that the maximum resistance R is obtained by choosing resistors for which $R_1 = R_2$

Soln : $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ where $R_1 + R_2 = a$ (constant)

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{a-R_1} = \frac{a}{R_1(a-R_1)} \Rightarrow R = \frac{R_1(a-R_1)}{a} = \frac{1}{a}(aR_1 - R_1^2)$$

$$R = \frac{1}{a}(aR_1 - R_1^2)$$

$$\frac{dR}{dR_1} = \frac{1}{a}(a - 2R_1)$$

For max or min $\frac{dR}{dR_1} = 0 \Rightarrow R_1 = \frac{a}{2}$

$$\frac{d^2R}{dR_1^2} = \frac{1}{a}(-2) = -\frac{2}{a} < 0$$

$\therefore$ R is maximum when $R_1 = \frac{a}{2}$ $\quad R_2 = \frac{a}{2}$

Hence $R_1 = R_2$.

9) A beam of length l is supported at one end, if w is the uniform load per unit length. The bending moment, M at a distance x from the end is given by $M = \frac{1}{2}lx - \frac{1}{2}wx^2$. Find the point on the beam at which the bending moment has the maximum value.

Soln: $$M = \frac{1}{2}lx - \frac{1}{2}wx^2$$

$$\frac{dM}{dx} = \frac{1}{2}l - \frac{1}{2}w2x = \frac{1}{2}l - wx$$

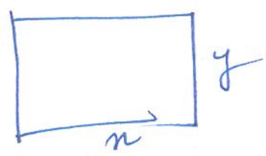
For max or min $\frac{dM}{dx} = 0 \Rightarrow x = \frac{l}{2w}$

$$\frac{d^2M}{dx^2} = -w < 0$$

Hence the beam has maximum bending moment at a point which is at a distance of $\frac{l}{2w}$ from one end.

10) Show that the area of a rectangle of given perimeter is maximum, when the rectangle is a square.

Soln: Let $P = 2x + 2y$ (say)



$\therefore y = \frac{1}{2}(p - 2x)$

$A = xy = x \times \frac{1}{2}(p - 2x)$

$= \frac{1}{2}(px - 2x^2)$

$\frac{dA}{dx} = \frac{1}{2}(p - 4x)$

For max. or min $\frac{dA}{dx} = 0 \Rightarrow x = \frac{p}{4}$

$\frac{d^2A}{dx^2} = \frac{1}{2}(-4) = -ve$

$\therefore$ area is maximum when $x = \frac{p}{4}$

$\therefore y = \frac{1}{2}(p - 2 \cdot \frac{p}{4}) = \frac{p}{4}$

Hence $\underline{x = y}$. It is a square.

11) Two sides of a triangle are given. Find the angle between them such that the area shall be maximum.

Soln: Let given sides be a and b and let θ be included angle.

$\therefore \Delta = \frac{1}{2} ab \sin\theta$ $\qquad \frac{d\Delta}{d\theta} = \frac{1}{2} ab \cos\theta$

for max or min $\frac{d\Delta}{d\theta} = 0 \Rightarrow \frac{1}{2} ab \cos\theta = 0 \Rightarrow \cos\theta = 0 \quad \theta = \frac{\pi}{2}$

$\frac{d^2\Delta}{d\theta^2} = -\frac{1}{2} ab \sin\theta$ $\qquad \left.\frac{d^2\Delta}{d\theta^2}\right)_{\theta = \frac{\pi}{2}} = -\frac{1}{2} ab \sin\frac{\pi}{2} = -\frac{1}{2}ab = -ve.$

$\therefore \Delta$ is maximum when $\theta = \frac{\pi}{2}$.

12) A closed right circular cylinder has a volume of 2156 cm^3. What will be the radius of its base so that its total surface area is minimum?

Soln: $V = \pi r^2 h = 2156 \text{ cm}^3$

$\therefore h = \dfrac{2156}{\pi r^2}$

Total Surface area $S = 2\pi r h + 2\pi r^2$

$$S = 2\pi r \frac{2156}{\pi r^2} + 2\pi r^2 = \frac{4312}{r} + 2\pi r^2$$

$$\frac{dS}{dr} = \frac{-4312}{r^2} + 4\pi r$$

For max or min $\dfrac{dS}{dr} = 0 \Rightarrow \dfrac{4312}{r^2} = 4\pi r$

$$\therefore r^3 = \frac{4312}{4\pi} = \frac{4312 \times 7}{4 \times 22} = 7^3 \Rightarrow r = 7 \text{ cm}.$$

$$\frac{d^2S}{dr^2} = \frac{2 \times 4312}{r^3} + 4\pi = \frac{8624}{r^3} + 4\pi$$

$$\left.\frac{d^2S}{dr^2}\right)_{r=7} = \frac{8624}{7 \times 7 \times 7} + 4\pi > 0$$

$\therefore S$ is minimum when $r = 7$ cm.

Hence for base radius $= 7$cm, the total surface area is minimum.