Chapter 1 - Relations and Functions

Definitions:

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

- **Relation**

  If $(a, b) \in R$, we say that a is related to b under the relation R and we write as a R b

- **Function**

  It is represented as $f: A \rightarrow B$ and function is also called mapping.

- **Real Function**

  $f: A \rightarrow B$ is called a real function, if A and B are subsets of R.

- **Domain and Codomain of a Real Function**

  Domain and codomain of a function f is a set of all real numbers x for which f(x) is a real number. Here, set A is domain and set B is codomain.

- **Range of a real function**

  f is a set of values f(x) which it attains on the points of its domain

Types of Relations

- A relation R in a set A is called **Empty relation**, if no element of A is related to any element of A, i.e., $R = \varphi \subset A \times A$.
- A relation R in a set A is called **Universal relation**, if each element of A is related to every element of A, i.e., $R = A \times A$.
- Both the empty relation and the universal relation are sometimes called **Trivial Relations**
- A relation R in a set A is called
  - **Reflexive**
    - if $(a, a) \in R$, for every $a \in A$,
  - **Symmetric**
    - If $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.
  - **Transitive**
    - If $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

- A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive
- The set E of all even integers and the set O of all odd integers are subsets of Z satisfying following conditions:
  - All elements of E are related to each other and all elements of O are related to each other.
  - No element of E is related to any element of O and vice-versa.
  - E and O are disjoint and $Z = E \cup O$.
  - The subset E is called the equivalence class containing zero, Denoted by [0].
  - O is the equivalence class containing 1 and is denoted by [1].

- **Note**
    - $[0] \neq [1]$
    - $[0] = [2r]$
    - $[1] = [2r + 1]$, $r \in Z$.

- Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets Ai called partitions or subdivisions of X satisfying:
    - All elements of Ai are related to each other, for all i.
    - No element of Ai is related to any element of Aj, $i \neq j$.
    - $\cup A_j = X$ and $A_i \cap A_j = \varphi$, $i \neq j$.
- The subsets A_i are called equivalence classes.

Note:

- Two ways of representing a relation
    - Roaster method
    - Set builder method
- If $(a, b) \in R$, we say that a is related to b and we denote it as ***a R b***.

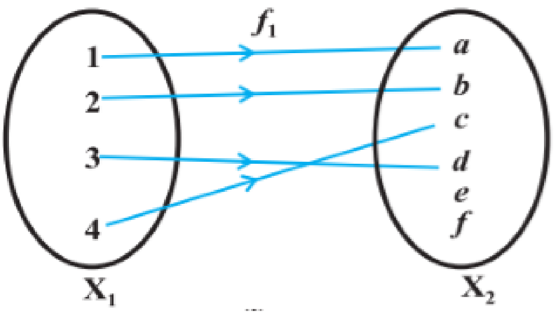
Types of Functions

Consider the functions f_1, f_2, f_3 and f_4 given

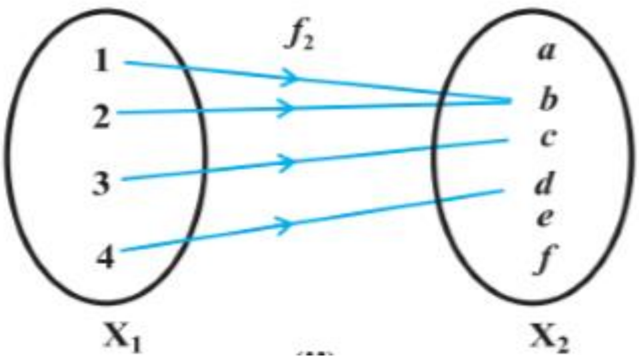
- A function f: $X \rightarrow Y$ is defined to be **one-one (or injective**), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called **many-one.**

Example

- One- One Function

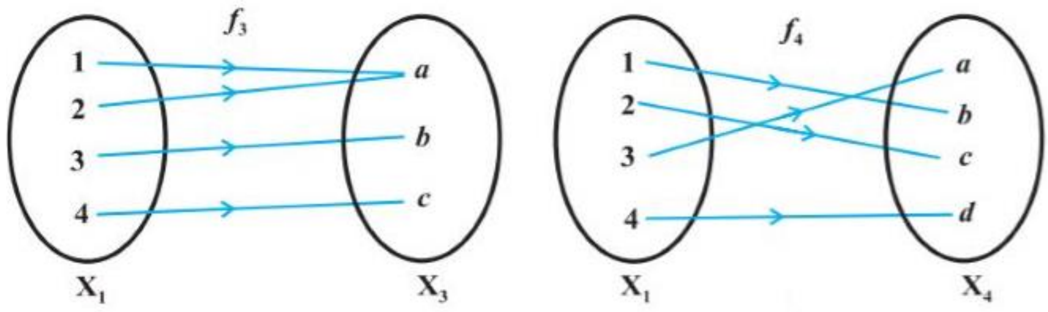


- Many-One Function

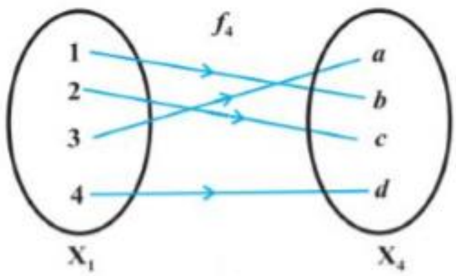


- A function f: $X \rightarrow Y$ is said to be **onto (or surjective)**, if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

- f: X → Y is onto if and only if Range of f = Y.
- Eg:



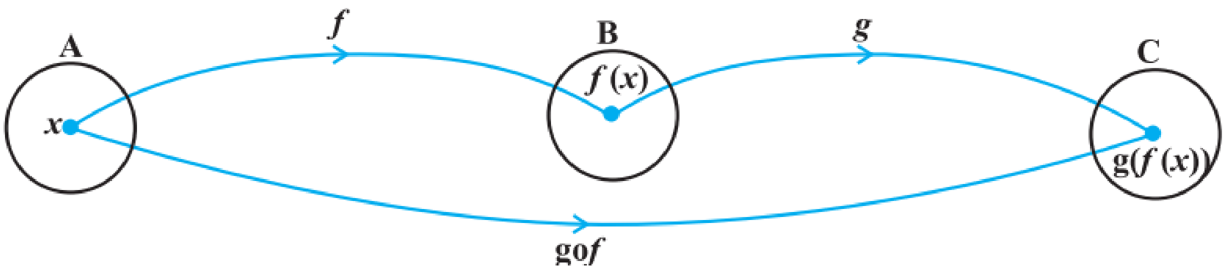
- A function f: X → Y is said to be **one-one and onto (or bijective),** if f is both one-one and onto.
  - Eg:



Composition of Functions and Invertible Function

Composite Function

- Let f: A → B and g: B → C be two functions.
- Then the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f$: A → C given by $g \circ f(x) = g(f(x)), \forall x \in A$.



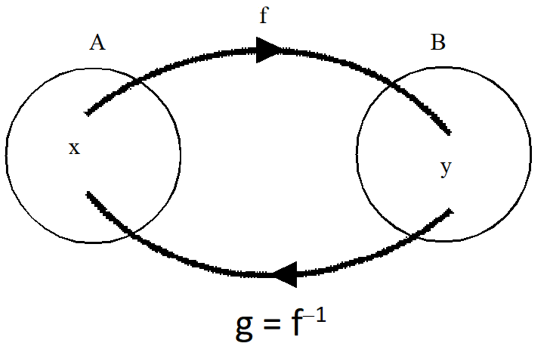
- Eg:
  - Let f : {2, 3, 4, 5} → {3, 4, 5, 9} and g : {3, 4, 5, 9} → {7, 11, 15} be functions
  - Defined as f (2) = 3, f (3) = 4, f (4) = f (5) = 5 and g (3) = g (4) = 7 and g (5) = g (9) = 11.
  - Find $g \circ f$.
  - Solution
    - $g \circ f(2) = g(f(2)) = g(3) = 7$,
    - $g \circ f(3) = g(f(3)) = g(4) = 7$,
    - $g \circ f(4) = g(f(4)) = g(5) = 11$ and
    - $g \circ f(5) = g(5) = 11$

- It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

- While composing f and g, to get gof, first f and then g was applied, while in the reverse process of the composite gof, first the reverse process of g is applied and then the reverse process of f.
- If $f: X \to Y$ is a function such that there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$, then f must be one-one and onto.

Invertible Function

- A function $f: X \to Y$ is defined to be invertible, if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f
- Denoted by f^{-1}.



- Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

Theorem 1

- If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then
  - $h \circ (g \circ f) = (h \circ g) \circ f$.
- Proof
  We have
  - $h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x)))$, $\forall$ x in X
  - $(h \circ g) \circ f(x) = h \circ g(f(x)) = h(g(f(x)))$, $\forall$ x in X.

  Hence, $h \circ (g \circ f) = (h \circ g) \circ f$

Theorem 2

- Let $f: X \to Y$ and $g: Y \to Z$ be two invertible functions.
  - Then gof is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
- Proof
  - To show that gof is invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, it is enough to show that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_X$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_Z$.
    Now, $(f^{-1} \circ g^{-1}) \circ (g \circ f) = ((f^{-1} \circ g^{-1}) \circ g) \circ f$, by Theorem 1
    $= (f^{-1} \circ (g^{-1} \circ g)) \circ f$, by Theorem 1
    $= (f^{-1} \circ I_Y) \circ f$, by definition of g^{-1}
    $= I_X$

  Similarly, it can be shown that $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_Z$

Binary Operations

Definitions:

- A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $*(a, b)$ by $a * b$.

- A binary operation $*$ on the set X is called commutative, if $a * b = b * a$, for every $a, b \in X$
- A binary operation $* : A \times A \rightarrow A$ is said to be associative if $(a * b) * c = a * (b * c)$, $\forall a, b, c, \in A$.
- A binary operation $* : A \times A \rightarrow A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall a \in A$.
    - Zero is identity for the addition operation on R but it is not identity for the addition operation on N, as $0 \notin N$.
    - Addition operation on N does not have any identity.
    - For the addition operation $+ : R \times R \rightarrow R$, given any $a \in R$, there exists $-a$ in R such that $a + (-a) = 0$ (identity for '+') $= (-a) + a$.
    - For the multiplication operation on R, given any $a \neq 0$ in R, we can choose $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$ (identity for '×') $= 1 = \frac{1}{a} \times a$
- A binary operation $* : A \times A \rightarrow A$ with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}