# Chapter 4 : Determinants (Class 12)

## Introduction

To each square matrix $A = [a_{ij}]$ of order $n$, there is associated a number (real or complex), which is called the determinant of square matrix $A$, where $a_{ij} = (i, j)$ th element in $A$.
This can be through as a function, which associates each square matrix with a unique number (real or complex).

Symbolically : $f : M \to K$
given by $f(A) = k$, where $M$ is the set of matrices and $K$ that of numbers (real or complex).
and $A \in M$ and $k \in K$.

Symbolically : $f(A) = |A| = \det A = \Delta = k$
$|A|$ or $\det A$ is read as determinant of $A$.

Only square matrices have determinants.

### (a) Determinant of a matrix of order one.

Let $A = [a]$ be a matrix of order $1 \times 1$

Then $\det A = |A| = |a| = a$.

Example, $A = [5]$, then $\det A = |5| = 5$.
$A = [-5]$, then $\det A = |-5| = -5$.
($|A|$ is not modulus of $A$ but is read as determinant of $A$).

### (b) Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$|A| = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Example, $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - (-1) \times 4 = 4 + 4 = 8$.

(c) Determinant of a matrix of order three

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example, Evaluate $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

$$\Delta = 1\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix}$$

$$= (-1\times 2 - 5\times 2) + 3(4\times 2 - 3\times 2) + 2(4\times 5 - 3\times(-1))$$

$$= (-2-10) + 3\times(8-6) + 2(20+3)$$

$$= -12 + 3\times 2 + 2\times 23$$

$$= -12 + 6 + 46$$

$$= -6 + 46$$

$$\Delta = 40$$

MINORS AND CO-FACTORS

Minors : The determinant obtained by deleting the ith row and jth column passing through the $a_{ij}$ is called the minor of element $a_{ij}$ in the determinant $|a_{ij}|$ of order n and is denoted by $M_{ij}$.

Co-factors: The co-factor of the element $a_{ij}$ is $(-1)^{i+j}$ time the determinant obtained by deleting the ith row and j column.

Co-factor of $a_{ij} = (-1)^{i+j} M_{ij}$.

Example,

consider $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of element $a_{21} = \begin{vmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ a_{21} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} = M_{21}$

$$= \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

Co-factor of element $a_{21} = A_{21} = (-1)^{2+1} \cdot M_{21} = - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

## Solved Examples

1) Write the co-factors of element of the second row of the determinants : $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ [ (1)]

2) Which of the following matrices are singular?

(i) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 10 \\ 1 & 0 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ [(i)]

[ HINT... : A square matrix A is said to be singular if $|A| = 0$.

3) Write the minors and co-factors of the elements of the 1st column of the following determinants and evaluate them:

(i) $\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -5 \\ 4 & -1 & -3 \end{vmatrix}$

## Properties of Determinants

(i) . The value of the determinant remains unaltered by interchanging its rows and columns (Reflection Property).
Now, let $\Delta'$ be the determinant which is formed from $\Delta$ by changing rows into columns or column into rows.

$$\Delta = \Delta'.$$

(ii) (Switching Property) If two adjacent rows (or columns) of a determinant are interchanged, then the sign of the determinant is changed.

$$\Delta' = -\Delta.$$

Corollary: If any row of a determinant is carried over $n$ rows, then $\Delta'$, the resultant determinant $= (-1)^n \Delta$.

(iii) (Repetition Property) If two rows (or columns) of a determinant are identical, then its value is zero.

(iv) (Scalar Multiple Property) If each element of a row (or column) of a determinant is multiplied by a scalar 'k', then its value gets multiplied by the scalar 'k'.

Corollary 1:
$$\begin{vmatrix} ka_1 & lb_1 & mc_1 \\ ka_2 & lb_2 & mc_2 \\ ka_3 & lb_3 & mc_3 \end{vmatrix} = klm \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Corollary 2:
$$\Delta = \begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0.$$

(v) If each element of one row or one column be the sum of two numbers, the determinant can be expressed as the sum of the two determinants of the same order.

$$\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

(vi) If equi-multiples of all elements of a row (or a column) are added to corresponding elements of any other row (or column), the determinant remains unaltered in value.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1+c_1 & b_1 & c_1 \\ a_2+mb_2+c_2 & b_2 & c_2 \\ a_3+mb_3+c_3 & b_3 & c_3 \end{vmatrix}$$

(vii) If those elements of a determinant $\Delta$ which involve $x$, are polynomial in $x$ and if $\Delta = 0$, when "$a$" be put for $x$ then $(x-a)$ is a factor of $x$.

(viii) a) Determinant of a skew-symmetric matrix of odd order is zero.

b) Determinant of a skew-symmetric matrix of even order is a perfect square

## Solved Examples

1) Evaluate (i) $\begin{vmatrix} 91 & 92 & 93 \\ 94 & 95 & 96 \\ 97 & 98 & 99 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 11 & 13 & 15 \end{vmatrix}$ [i) Ans. 0]. ii) Ans. 0]

3) Evaluate $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ (Ans 0)

4) Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (Ans. 0).

5) Using properties of determinants and without expanding, prove that $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0.$

6) Without expanding the determinant, prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

7) Without expanding the determinant, show that $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & cb \end{vmatrix} = 0$

8) Find the value of $\theta$ if $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ [ 1/2 ]

9) Without expanding the determinants, show that

$$\begin{vmatrix} 2 & 1 & -7 \\ -4 & -3 & 8 \\ 6 & 5 & -9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 8 \\ 3 & 5 & 9 \end{vmatrix}$$

10) Without expanding prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

11) Without expanding the determinant, evaluate $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$ [0]

12) If $A+B+C=\pi$, prove that $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix} = 0.$

13) Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

14) Prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0.$

15) Find the value of determinant $\begin{vmatrix} \sqrt{x}+\sqrt{y} & 2\sqrt{2} & \sqrt{z} \\ \sqrt{yz}+\sqrt{2}x & z & \sqrt{2}z \\ y+\sqrt{xz} & \sqrt{yz} & z \end{vmatrix}$

[Ans. $z(\sqrt{2}y - z\sqrt{y})$]

16) If $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$

17) Prove that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$

18) Prove that $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

19) Show that $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3+b^3+c^3-3abc.$

20) Prove that $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

21) Show without expanding the determinant $\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$

22) Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

23) Let $\Delta = \begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$ Show that $\Delta$ is a negative integer.

24) Show that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

25) If $a, b, c$ are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$

Show that $abc(ab+bc+ca) = a+b+c$

26) Evaluate $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$ $[4a^2b^2c^2]$

## Using Factor Theorem for Evaluating Determinants

1) If a determinant $\Delta$ becomes zero by putting $x=\alpha$ in it, then $(x-\alpha)$ is a factor of $\Delta$.

2) If on putting $a=b$ in the determined $\Delta$, any two of its rows or columns become identical, then $\Delta=0$.

3) If on putting $a=b$ in $\Delta$, $p$ rows or columns becomes identical, then $(a-b)^{p-1}$ is a factor of $\Delta$.

Example 1: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Soln: If we put $a=b$, first two columns become identical.
$\therefore$ $\Delta$ becomes zero and $(a-b)$ must be a factor of $\Delta$.
Similarly, $(b-c)$ and $(c-a)$ must also be factors of $\Delta$.
Thus, $(a-b)(b-c)(c-a)$ is a factor of $\Delta$ of degree 3.

Also $\Delta$ is of degree 3 in $a, b, c$ because deg $(1 \cdot b \cdot c^2)$ is 3.

$\therefore$ we have the identity

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = k(a-b)(b-c)(c-a)$$

where $k$ is a constant.
Putting $a=0$, $b=1$, $c=2$ in this identity, we have
$k=1$.

$\therefore$ $\Delta = (a-b)(b-c)(c-a)$.

Example 2: Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$

Soln:
$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Taking $x$, $y$ and $z$ common from $C_1$, $C_2$ & $C_3$.

$$\Delta = xyz \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

If put $x = y$, $y = z$ and $z = x$

We get $\Delta = 0$

$\therefore$ $(x-y)(y-z)(z-x)$ is factor $\Delta$.

$\therefore$ $\Delta = xyz(x-y)(y-z)(z-x)$

Example 3: Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

Soln: Let
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Putting $a=b$ in $\Delta$, $\Delta$ becomes zero. $\therefore$ $(a-b)$ is factor.
Similarly $(b-c)$ and $(c-a)$ is factor of $\Delta$.
Thus $(a-b)(b-c)(c-a)$ is factor of $\Delta$ in degree 3.

But $\Delta$ is of degree 4 in $a$, $b$ and $c$. $(1 \cdot b \cdot c^3) = 4$.

$\therefore$ $\Delta$ must have factor of first degree and it should be symmetrical in $a, b, c$.

Let it be $k(a+b+c)$, where $k$ is a constant.

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = k(a-b)(b-c)(c-a)(a+b+c)$$

Putting $a=0$, $b=1$ and $c=2$, we get

$$k = 1.$$

$$\therefore \quad \Delta = (a-b)(b-c)(c-a)(a+b+c)$$

__Example 4__: Prove that $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)$

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

If we put $x=y$, row $R_1$ & $R_2$ become identical

$\therefore$ $(x-y)$ is factor of $\Delta$

Similarly, $(x-y)(x-y)(z-x)$ are factor of $\Delta$ in degree 3.

As the determinant is symmetrical and of $5^{th}$ degree,
So the remaining factor must be quadratic in $x$, $y$ and $z$.

Let it be $[h(x^2+y^2+z^2)+k(xy+yz+zx)]$

$$\therefore \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (y-z)(z-x)(x-y)\left[h(x^2+y^2+z^2)+k(xy+yz+zx)\right]$$

Putting $x=0$, $y=1$ and $z=2$, we get

$$5h+2k=2$$

Again Putting $x=0$, $y=2$ and $z=3$, we get

$$13h + 6k = 6$$

Solving
$$5h + 2k = 2$$
$$13h + 6k = 6$$
we get $h=0$ and $k=1$

$$\therefore \Delta = (y-z)(z-x)(x-y)(xy+yz+zx).$$

<u>Example 5</u>: Prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$

If we put $a=b$ in $\Delta$, $C_1$ & $C_2$ become identical.
$\therefore (a-b)$ is factor of $\Delta$.

Similarly $(b-c)$ & $(c-a)$ is factor of $\Delta$
$\therefore (a-b)(b-c)(c-a)$ is factors of $\Delta$ in degree 3.

But $\Delta$ is of degree 4 in $a, b, \& c$. $\Delta$ must have one more factor of first degree and it should be symmetric in $a, b, c$. Let it be $k(a+b+c)$, where $k$ is a constant.

$$\therefore \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = k(a-b)(b-c)(c-a)(a+b+c)$$

Putting $a=0$, $b=1$ and $c=2$ we get $k=1$
$$\therefore \Delta = (a-b)(b-c)(c-a)(a+b+c).$$

## Solving Determinant Equation

**Example 1:** Solve the equation $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$

Sol$^n$:

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, $\begin{vmatrix} x+2 & 1 & 1 \\ x+2 & x & 1 \\ x+2 & 1 & x \end{vmatrix} = 0$

$$(x+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$(x+2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} = 0$$

$$(x+2)(1)\begin{vmatrix} x-1 & 0 \\ 0 & x-1 \end{vmatrix} = 0 \Rightarrow (x+2)(x-1)^2 = 0$$

$$\Rightarrow x = -2 \text{ or } 1.$$

**Example 2:** Solve the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix}$

Sol$^n$:

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, $\begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix} = 0$

$$(3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1 \qquad R_3 \to R_3 - R_1$

$$(3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix} = 0$$

$$(3x-2)(1)\begin{vmatrix} 3x-11 & 0 \\ 0 & 3x-11 \end{vmatrix} = 0 \Rightarrow (3x-2)(3x-11)^2 = 0$$

$$\therefore\ x = \frac{2}{3} \text{ or } \frac{11}{3}$$

**Example 3:** If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$ is $x = -9$, then find the other two roots.

Sol$^n$ Given $\begin{vmatrix} 7 & 6 & x \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$, $\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$

$$(x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$$

Applying $C_1 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$(x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ x & x-3 & 7-x \end{vmatrix} = 0, \quad (x+9)\begin{vmatrix} x-2 & 0 \\ x-3 & 7-x \end{vmatrix}$$

$$(x+9)(x-2)(7-x) = 0 \Rightarrow x = -9, 2, 7.$$

# Application of Determinants to Coordinate Geometry

## Area of Triangle

Area of triangle having vertices at $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since the area has to be positive, take the absolute value of the determinant.

## Condition of Collinearity of Three Points

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if the area of $\Delta$ is zero.

$$\therefore \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 1 : Find the area of triangle with vertices

(i) $(0,0)$, $(6,0)$, $(4,3)$ [ 9 ]

(ii) $(2,7)$, $(1,1)$, $(10,8)$ [ $\frac{47}{2}$ ]

Example 2 : Find whether following points are collinear.
$(11,7)$, $(5,5)$ and $(-1,3)$ [Collinear].

## Solutions of System of Linear Equations

### Cramer's Rule for System in Two variables:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

If $D \neq 0$, then solution is given by

$$x = \frac{D_1}{D} \qquad y = \frac{D_2}{D}$$

### Cramer's Rule for System in Three Variables:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D} \quad \text{and} \quad z = \frac{D_3}{D}$$

### System of three simultaneous linear non-homogeneous equations in three unknowns $x$, $y$ and $z$.

(i) It is consistent and has unique solution which is given by
$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D} \quad \text{if } D \neq 0.$$

(ii) It is consistent and has infinitely many solutions if $D = 0$ and $D_1 = D_2 = D_3 = 0$

(iii) It is inconsistent (i.e. no solutions) if $D = 0$ and at least one of the determinants $D_1$, $D_2$ and $D_3$ is non-zero.

## System of homogeneous linear equation in three variables.

If $d_1 = d_2 = d_3 = 0$, then system of linear equation is known as homogeneous system of equations.

Now $D_1 = D_2 = D_3 = 0$ (If one column is zero) the value of determinent is zero.

(i) If $D \neq 0$, the system has only the trival solutions which is $x = y = z = 0$.

(ii) If $D = 0$, the system has non-trival (non zero) solution. In this case infinite number of solution exist.

## Examples

1) Solve the following: (i) $\begin{aligned} 6x + y - 3z &= 5 \\ x + 3y - 2z &= 5 \\ 2x + y + 4z &= 8 \end{aligned}$ (ii) $\begin{aligned} x + y - z &= 0 \\ x - 2y + z &= 0 \\ 3x + 6y - 5z &= 0 \end{aligned}$

$(x = 1, y = 2, z = 1)$ $\left(x = \frac{1k}{3}, y = \frac{2k}{3}, z = k\right)$

2) The sum of three numbers is 6. If we multiply the third by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times of the first numbers, we get 12. Find the numbers. (3, 1, & 2).

# Adjoint of a Matrix

The adjoint of a square matrix $A = [a_{ij}]$ is the transpose of matrix obtained by replacing each element of $A$ by its co-factor in $|A|$.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

$$\text{Adj. } A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Where $A_{ij}$ denotes the co-factor.

**Theorem:**

If $A$ be any $n$-rowed square matrix, then $(\text{Adj } A)\, A = A\,(\text{Adj } A) = |A|\, I_n$

The theorem states that the matrices $A$ and Adj $A$ are commutative and that their product is a scalar matrix every diagonal element of which is $|A|$.

$$A\,(\text{Adj } A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & |A| \end{bmatrix} = |A|\, I_n .$$

(i) $\text{Adj } A' = (\text{Adj } A)'$

(ii) $\text{Adj}\,(AB) = (\text{Adj}\, B)\,(\text{Adj}\, A)$.

## Solved Examples

1) Show that adjoint of the matrix $N = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is $N$.

2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$, find $A(\text{Adj } A)$ with finding adj. $A$.

[ Hint.: $A(\text{Adj } A) = |A| I_3$ ]. (Ans. $-14\, I_3$)

3) Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ and verify that $A(\text{Adj}\cdot A) = |A| I_3$. Ans. $\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & 1 \end{bmatrix}$

4) If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $A(\text{Adj}\cdot A) = |A| I$.

# Inverse of Matrix

**Singular Matrix:** A square matrix A is called a singular matrix or a non-singular matrix as $|A| = 0$ or $|A| \neq 0$.

If $|A| = 0$, Matrix singular.

If $|A| \neq 0$, matrix non-singular matrix.

## Inverse of a Square Matrix

The inverse of an $n \times n$ matrix A is an $n \times n$ matrix $A^{-1}$ (if exists) such that $AA^{-1} = I$ and $A^{-1}A = I$, read A inverse.

If A has an inverse, then A is invertible.
The inverse of a square matrix if it exists in unique.

**Theorem:** The necessary and sufficient condition for a square matrix A to posses inverse is that $|A| \neq 0$ i.e. A is non-singular.

(i) $AB = BA = I$, $|A| \neq 0$.

(ii) $AB = BA = I$, $A^{-1} = B = \dfrac{\text{Adj } A}{|A|}$

**Theorem:** If A, B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

$$(A^T)^{-1} = (A^{-1})^T$$

For Example

1) To check whether the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ exist.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$|A| = 1(1) - 0 + 1(4-3) = 1+1 = 2$$

$$|A| = 2 \neq 0$$

We know that necessary and sufficient condition for a square matrix A to be invertible is $|A| \neq 0$.

Hence, the inverse of the matrix exist.

2) To determine whether $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ are inverse of each other.

Find the product of AB and BA.

$$AB = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 21-20 & -12+12 \\ 35-35 & -20+21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 21-20 & 28-28 \\ -15+15 & -20+21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore AB = BA = I$$

$\therefore$ A and B are inverse of each other.

3) To find the inverse of $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} \qquad |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 - (-15)$$

$$|A| = -2 + 15 = 13 \neq 0$$

$\therefore A^{-1}$ exist and is given by

$$A^{-1} = \frac{\text{Adj. } A}{|A|.}$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{13} \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}.$$

## Solved Examples.

1) Find the inverse of the diagonal matrix diag. $[a \ b \ c] \neq 0$.

Ans. Diag $\left[\frac{1}{a} \ \frac{1}{b} \ \frac{1}{c}\right]$

2) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find $A^2$ and show that $A^2 = A^{-1}$.

3) If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A' A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

4) For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find $x$ and $y$ so that $A^2 + xI = yA$. Hence find $A^{-1}$

$A^{-1} = \frac{1}{8} \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$ $\quad x = 8$, $y = 8$.

## Consistency and Inconsistency of a System of Equations.

(i) If $|A| \neq 0$, then the system is consistent and has a unique solution.

(ii) If $|A| = 0$, the system of equation has either no solutions or an infinite number of solution.

Now find $(\text{Adj } A)B$

(a) If $(\text{Adj } A)B \neq 0$, the system is inconsistent, has no solution.

(b) If $(\text{Adj } A)B = 0$, the system is consistent and has many $\text{sol}^n$.

**Example:** Use matrix method to examine the following system of equation for consistency or inconsistency and solve.

1) $3x - 2y = 5$, $6x - 4y = 9$. [No solution]

2) $4x - 2y = 3$, $6x - 3y = 5$ [No solution]

3) $6x + 4y = 2$, $9x + 6y = 3$ [Consistent, $y = k$, $x = \frac{1}{3}(1-2k)$]

4) $4x - 5y - 2z = 2$
$5x - 4y + 2z = -2$
$2x + 2y + 8z = -1$ [No solution]

5) $5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 9$ [infinite solution, $x = \frac{1}{11}(7 - 16k)$, $y = \frac{1}{11}(k+3)$, $z = k$.]

6) $x - y + z = 3$
$2x + y - z = 2$
$x + 2y - 2z = -1$ [consistent, many solution, $x = \frac{5}{3}$, $y = k - \frac{4}{3}$ $z = k$.]

## System of Linear Equations : Solving a System of Linear Equations by Matrix Method

The system of equations given can be written in matrix form as

$$AX = B$$

A – coefficient matrix

X – column matrix of unknowns

B – column matrix of given constants.

<u>Theorem</u>: If $A$ is non-singular, then $X = A^{-1}B$ is the unique solution of the matrix $AX = B$.

$$X = A^{-1}B \qquad A^{-1} = \frac{Adj\,A}{|A|}$$

<u>Examples</u> :

1) Solve the following system of equations by matrix.
$5x+2y=4$ , $7x+3y=5$
[$x=2$ , $y=-3$]

2) Solve the following system of equation :
$x+2y-3z=6$ ; $3x+2y-2z=3$ ; $2x-y+z=2$ [$x=1, y=-5, z=-5$]

3) Solve the following equation : $\frac{2}{x}-\frac{3}{y}+\frac{3}{z}=10$ ; $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10$
$\frac{3}{x}-\frac{1}{y}+\frac{2}{z}=13$ .
($x=\frac{1}{2}$ , $y=\frac{1}{3}$ , $z=\frac{1}{5}$).

4) If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ , find $A^{-1}$. Using $A^{-1}$, solve the system of linear equation :

$$x-2y=10$$
$$2x+y+3z=8$$
$$-2y+z=7$$

[$x=4$, $y=-3$, $z=1$]

# Solution of System of Homogeneous Linear Equations.

A linear equation of the type

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = 0$$

is called a homogeneous linear equation.

The equations
$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

constitute a system of homogeneous linear equations.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then the system of equations can be written as

$$AX = 0.$$

$x = y = z = 0$ is always a solution and is called the trivial solution, when $A$ is non-singular. ($|A| \neq 0$).

Theorem: A system of $n$ homogeneous linear equation in $n$ unknowns has non-trivial solutions if and only if the coefficient matrix is singular.

## Summary:

When system of equations is homogeneous i.e. $AX = 0$. $A$ is square.

(i) If $|A| \neq 0$ ($A$ is non-singular), the system has only trivial solution. It has one solution $x = y = z = 0$

(ii) If $|A| = 0$, the system of equations has infinitely many solutions (both trivial i.e., all zero and non-trivial solutions.)

## Examples

1) Solve $2x+3y-z=0$; $x-y-2z=0$ $3x+y+3z=0$. [Trivial soln]

2) Solve $x+y-2z=0$; $2x+y-3z=0$ $5x+4y-9z=0$.
[Trivial soln]

3) Solve $x+y-2z=0$, $2x+y-3z=0$, $5x+4y-9z=0$

[Non-trivial soln].
$x=y=z=k$.

4) Solve $x+y-z=0$, $x-2y+z=0$, $3x+6y-5z=0$.

[Non-trivial soln,
$x=k$, $y=2k$, $z=3k$].