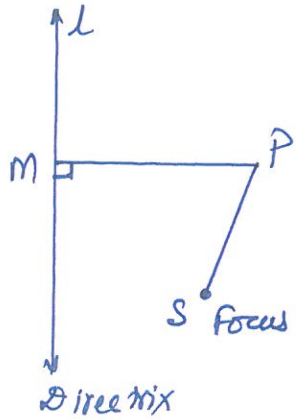
Chapter 11 - CONIC SECTIONS

1) A conic section is a curve in a plane that can be obtained by the intersection of a double-napped right circular cone with a plane.

The three types of conic section are the hyperbola, the parabola, and the ellipse; the circle is a special case of the ellipse, though sometimes called a fourth type.

2) <u>Definition</u> : Conic section section is the set of all points P in a plane such that its distance from a fixed point (S) and a fixed line (l) not containing the fixed point are in a constant ratio.



The fixed point is called the focus and is generally denoted by S.

The fixed line is called the directrix and is denoted by l.

By definition $\frac{SP}{PM}$ = constant

This constant ratio is called the <u>eccentricity</u> of the conic section and is generally denoted by the letter 'e'.

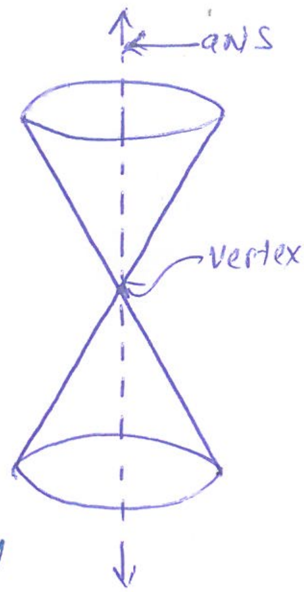
3) (i) If $e=1$, the conic section is called a parabola

(ii) If $0<e<1$, the conic section is called an ellipse.

(iii) If $e=0$, the conic section is called a circle

(iv) If $e>1$, the conic section is called a hyperbola.

4) Definitions :



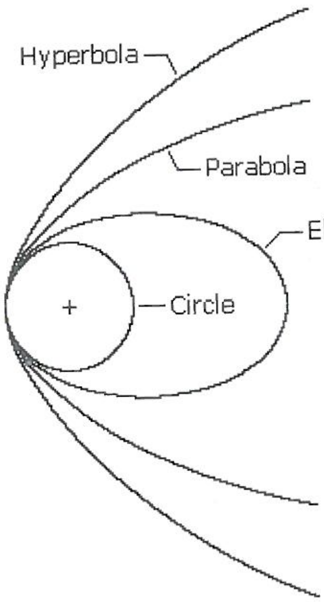
a) The axis about which the conic section is symmetrical is called the axis of the conic section.

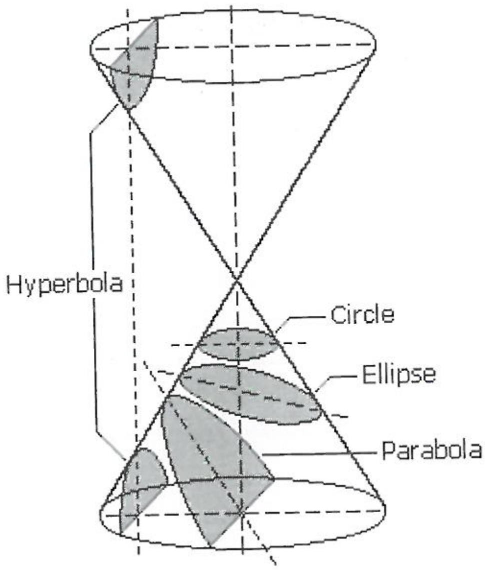
b) The point of intersection of the conic section with its axes is called the vertex of the conic section.

c) Focal chord : A line segment joining any two points on the conic section is called a chord and a chord of a conic section passing through its focus is called a focal chord.

d) Focal Distance : The distance of a point on a conic from the focus is called the focal distance of the point.

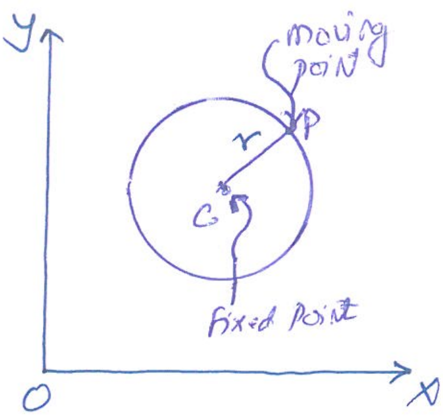
e) Latus Rectum : The latus rectum of a conic is the chord through the focus and perpendicular to the axis.





5) The Circle

Definition: A circle is the set of all points in a plane which are at a constant distance from a fixed point in the plane.



The fixed point is called the centre and the given distance is called the radius of the circle.

Standard form of the Equation of a Circle:

Let $C(h,k)$ be the centre of the circle.
Let $P(x,y)$ be any point on the circle
Let r be the radius of the circle.
By definition $|CP| = r$

Using distance formula $|CP| = \sqrt{(x-h)^2 + (y-k)^2} = r$

$$\therefore \quad (x-h)^2 + (y-k)^2 = r^2$$

If h and k are zero, the equation of the circle with centre at origin $O(0,0)$ and radius r is

$$(x-0)^2 + (y-0)^2 = r^2$$

$$\therefore \quad x^2 + y^2 = r^2$$

This is called the standard equation of circle.

A point circle is the circle whose radius is zero.

6) General Equation of a circle

The general equation of a circle is of the form

$$x^2+y^2+2gx+2fy+c=0$$

centre $(-g,-f)$ radius $=\sqrt{g^2+f^2-c}$

(i) If $g=0$, then centre of circle $=(0,-f)$ lies on y-axis.

(ii) If $f=0$, the centre of circle $=(-g,0)$ lies on x-axis.

(iii) If $c=0$, then the equation of the circle reduces to

$$x^2+y^2+2gx+2fy=0.$$

the circle passes through origin.

Circles having the same centre are called concentric circles.
for example, $x^2+y^2+2gx+2fy+c_1=0$
$x^2+y^2+2gx+2fy+c_2=0$.

Solved Examples

(1) Find the equation of the circle whose centre is $(2,-3)$ and passing through the intersection of the lines $3x-2y=1$ and $4x+y=27$.
$[x^2+y^2-4x+6y-96=0]$.

(2) Find the equation of a circle of radius 5 whose centre lies on x-axis and which passes through the point $(2,3)$.
$[x^2+y^2+4x-21=0]$.

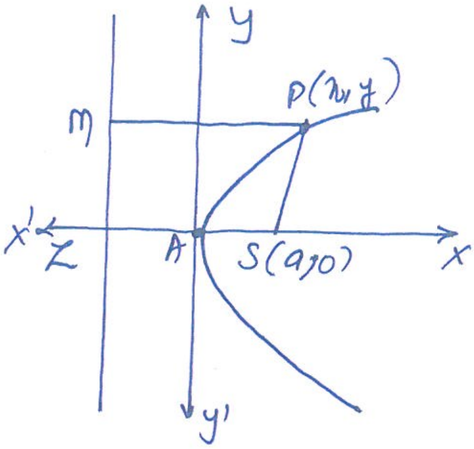
(3) Find the equation of the circle which passes through the centre of circle $x^2+y^2+8x+10y-7=0$ and is concentric with circle $2x^2+2y^2-8x-12y-9=0$. $[x^2+y^2-4x-6y-87=0]$.

(4) Find the equation(s) of the circle passing through the points $(1,1)$ and $(2,2)$ and whose radius is 1. $[x^2+y^2-2x-4y+4=0$
$x^2+y^2-4x-2y+4=0]$

7) The Parabola

Definition: A parabola is the set of all points in a plane equidistant from a fixed point and a fixed line in that plane.

Equation of a parabola in the standard form $y^2 = 4ax$.



This is the equation of the parabola in standard form whose vertex is at the origin, focus at $(a,0)$.

8) Four standard forms of the Parabola:

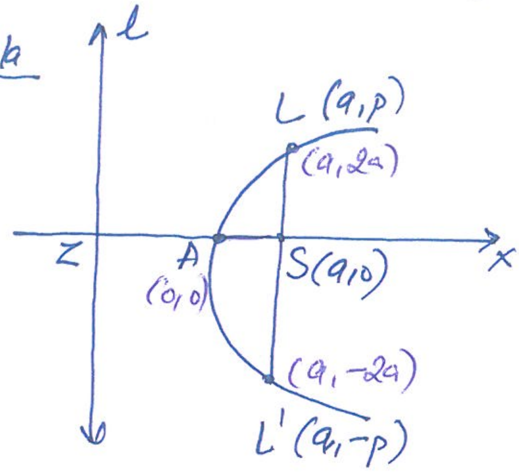
(i) $y^2 = 4ax$ (ii) $y^2 = -4ax$ (iii) $x^2 = 4ay$ (iv) $x^2 = -4ay$

9) Length of the latus rectum of the parabola

LSL' – latus rectum,

(The latus rectum of a parabola is the chord passing through the focus and is perpendicular to the axis.) .

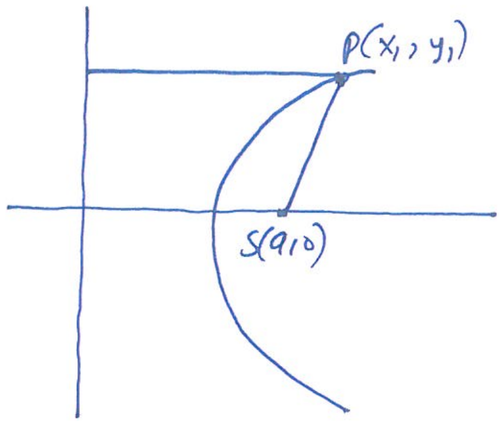
let $LS = p \therefore = \pm 2a$



The length of the latus rectum of a parabola is four times the distance of its vertex from the focus.

In each parabola length of the latus rectum $= 4a$.

10) Focal distance (PS) of any point $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $a + x_1$



length of latus rectum

$= 2 \times$ length of $\perp$ from focus on directrix.

11) $(y-\beta)^2 = 4a(x-\alpha)$
It represents a parabola whose axis is parallel to x-axis. Vertex (α, β) and focus $(a+\alpha, \beta)$.

It can be reduced to the form $x = Ay^2 + By + C$ which is the equation of a parabola with axis parallel to x-axis.

12) $(x-\alpha)^2 = 4a(y-\beta)$
It represents a parabola whose axis is parallel to y-axis. Vertex is (α, β). focus $(\alpha, a+\beta)$

It can be reduced to the form $y = Ax^2 + Bx + C$ which is the equation of the parabola with axis parallel to y-axis and vertex (α, β).

Solved Examples

1) Find the standard equation of the parabola having its
(i) focus at $(2,0)$ and the directrix $x+2=0$ [$y^2 = 8x$]
(ii) focus at $(6,0)$ and the directrix $x=0$ [$y^2 = 12(x-3)$]

2) Find the equation of the parabola whose focus is $(3,-4)$ and directrix is the line $x+y-2=0$.
Also find the latus rectum and the equation of axis.
[$x^2+y^2-2xy-8x+20y+46=0$, $3\sqrt{2}$, $x-y-7=0$.]

3) Find the latus rectum, co-ordinates of the focus, equation of the directrix and co-ordinates of ends of the latus rectum of each of the following parabolas:
(i) $y^2 = 10x$ [$4a$, $(5/2, 0)$, $2x+5$, $(\frac{5}{2}, 5)$, $(\frac{5}{2}, -5)$]

4) Find the equation to the parabola whose focus is $(1,-1)$ and whose vertex is $(2,1)$. [$4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$]

5) Find the co-ordinates of the focus and the vertex, the equation of the axis, directrix and tangent at vertex for the parabola $x^2+4x+4y+16=0$. [$(0,-1)$, $(0,0)$, $x=0$, $y=a=1$, $y=0$,

14) The Ellipse

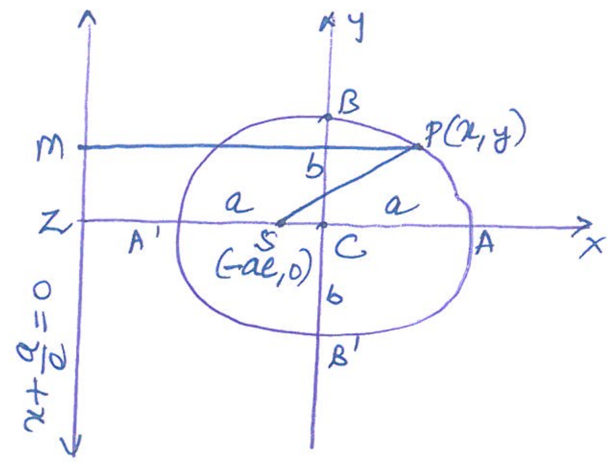
Definition: An ellipse is the set of all points in a plane whose distances from a fixed point in the plane and their distances from a fixed line in that plane bear a constant ratio, less than one.
The fixed point is called focus, the fixed line directrix and the constant ratio ($e < 1$), the eccentricity of the ellipse.

Equation of an ellipse in the standard form : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{ZA}{AS} = \frac{1}{e}$ and $\frac{ZA'}{A'S} = \frac{1}{e}$

$AS = eZA$ $\quad A'S = eZA'$

$\frac{SP}{MP} = e < 1$ (for Ellipse)

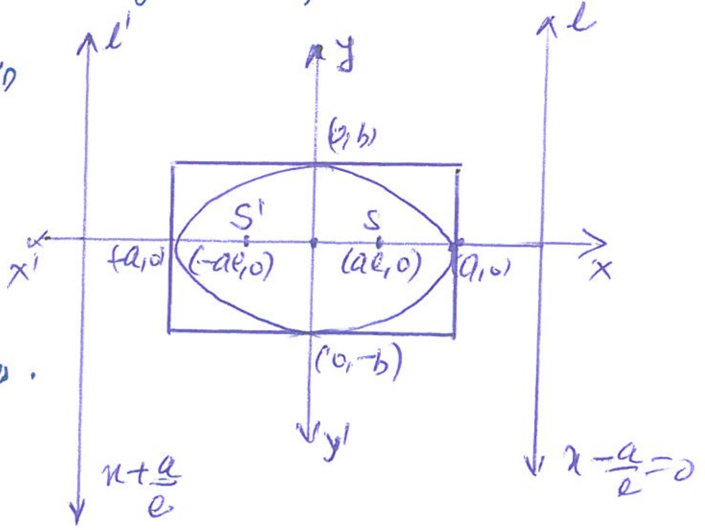


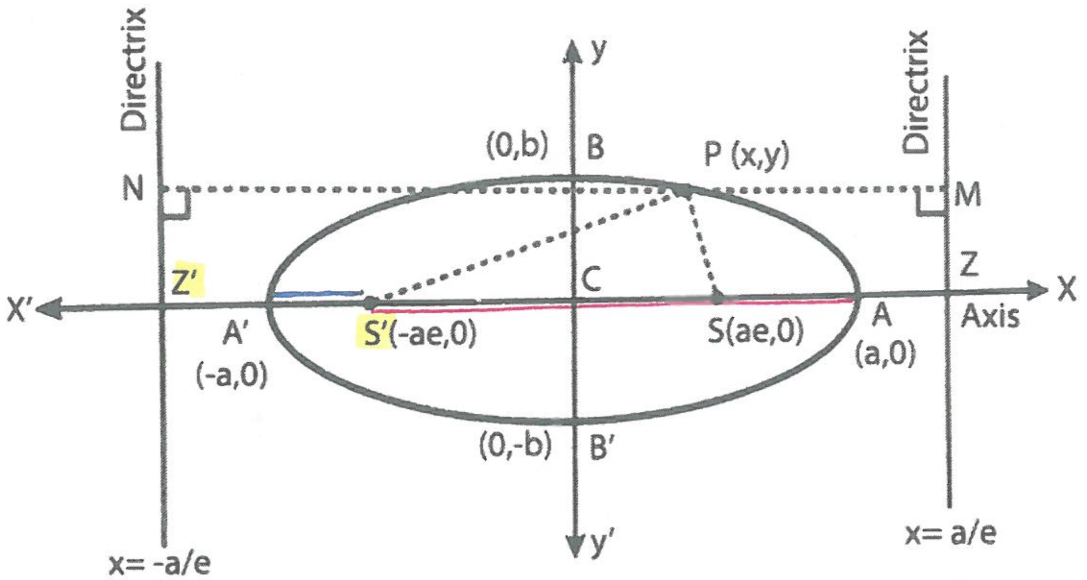
15) The ellipse is symmetrical about x-axis as well as about y-axis. Also the ellipse is symmetrical about the origin. Thus, origin is the centre of the ellipse.

16) An ellipse has two axes. $x = \pm a$ $\quad y = \pm b$
AA' is called the major axis
BB' is called the minor axis of the ellipse.

17) The ellipse lies completely within the rectangle formed by lines $x = \pm a$, $y = \pm b$.

18) The symmetry of the ellipse has two foci and two directrices.





Since $e<1$, divide $Z'S'$ both internally and externally in the ratio $1:e$ $\therefore$ $\frac{Z'A}{AS'} = \frac{1}{e}$ and $\frac{Z'A'}{A'S'} = \frac{1}{e}$

$AS' = e\,Z'A$ —(1) and $A'S' = e\,Z'A'$ —(2)

Thus, by definition A and A' lies on ellipse $\frac{PS'}{PN} = e < 1$

Let $AA' = a$ and $BB' = b$ and C be centre of AA' and BB'

$\therefore$ $CA = CA' = a$

Adding (1) & (2) $AS' + A'S' = e(Z'A + Z'A')$

$AS' + A'S' = e[CZ' + \cancel{CA} + CZ' - \cancel{CA'}]$

$2a = e(2CZ') = e\,2CZ'$ $\Rightarrow$ $CZ' = \frac{a}{e}$ —(3)

$\downarrow$ gives

Equation of $Z'N$

$x + \frac{a}{e} = 0$

Subtracting (2) from (1)

$AS' - A'S' = e(Z'A - Z'A')$

$(CS' + CA) - (CA' - CS') = e\,AA'$

$CS' + \cancel{CA} - \cancel{CA'} + CS' = e\,AA'$

$2CS' = e\,AA' = e\,2a$

$CS' = ae$ $\rightarrow$ gives coordinate of $S'(-ae, 0)$

Standard Equation

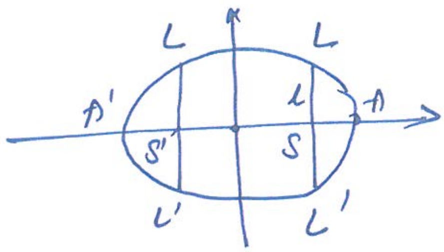
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

> If the centre of ellipse is at point (h,k) and axes are parallel

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

19) Length of Latus Rectum of Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



LSL' = latus rectum

let $SL = l$ $\quad S(ae, 0)$

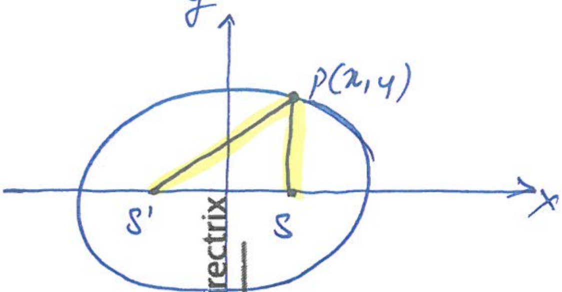
$\therefore L(ae, l)$

Point (ae, l) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ length of latus rectum $= 2l = \frac{2b^2}{a}$

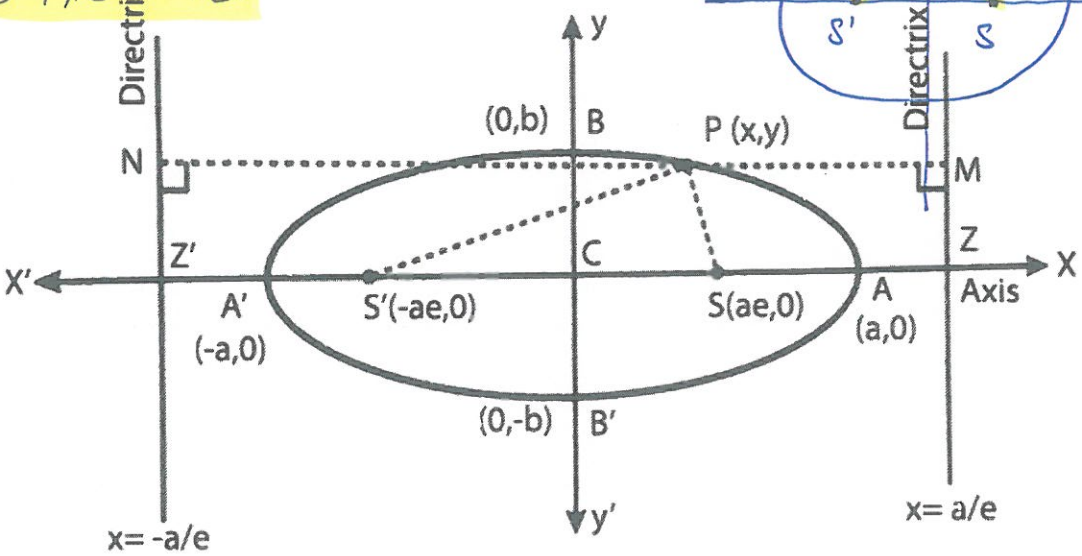
20) Focal Property of the Ellipse:

The sum of the focal distance of any point on an ellipse is constant and is equal to the length of the major axis of the ellipse.



focal distance $= PS = PS'$

$PS + PS' = 2a$



Focus of standard ellipse is $(ae, 0)$

$b^2 = a^2(1-e^2)$ or $e^2 = 1 - \frac{b^2}{a^2}$

Solved Examples

(1) Obtain the equation of an ellipse whose focus is the point $(-1,1)$, whose directrix is the $x-y+3=0$ and whose eccentricity is $1/2$. $[7x^2+7y^2+2xy+10x-10y+7=0]$

2) If the focus of a standard ellipse is at $(1,0)$ and corresponding directrix has the equation $x=4$, find the equation.
$[x^2/4 + y^2/3 = 1]$

(3) Find the coordinates of the foci, the vertices, the length of major axis, the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$

[major axis $= 12$, minor axis $= 8$, $e=\frac{\sqrt{20}}{6}$ foci $(\pm\sqrt{20},0)$
vertices $(\pm 6,0)$, latus rectum $=\frac{16}{3}$].

(4) Find the equation of ellipse when the co-ordinates of its vertices are $(\pm 5,0)$ and foci are $(\pm 4,0)$. $[9x^2+25y^2=225]$.

5) Find the equation of the ellipse whose axis are along the axes of co-ordinates and centre at the origin of co-ordinates:
(i) where major axes $= 8$ and $e=\frac{1}{2}$ $[3x^2+4y^2=15]$

(ii) where latus rectum $= 8$ and $e=\frac{1}{\sqrt{2}}$ $[x^2+2y^2=64]$

6) Find the equation of ellipse whose ends of major axis is $(\pm 3,0)$, ends of minor axis is $(0,\pm 2)$. $[\frac{x^2}{9}+\frac{y^2}{4}=1]$

7) Find the equation of ellipse whose length of minor axis is 16 and foci is $(0,\pm 6)$. $[\frac{x^2}{64}+\frac{y^2}{100}=1]$

The Hyperbola

Definition: A hyperbola is the set of all points in a plane whose distances from a fixed point in the plane and their distances from a fixed line in the plane bear a constant ratio, greater than one.

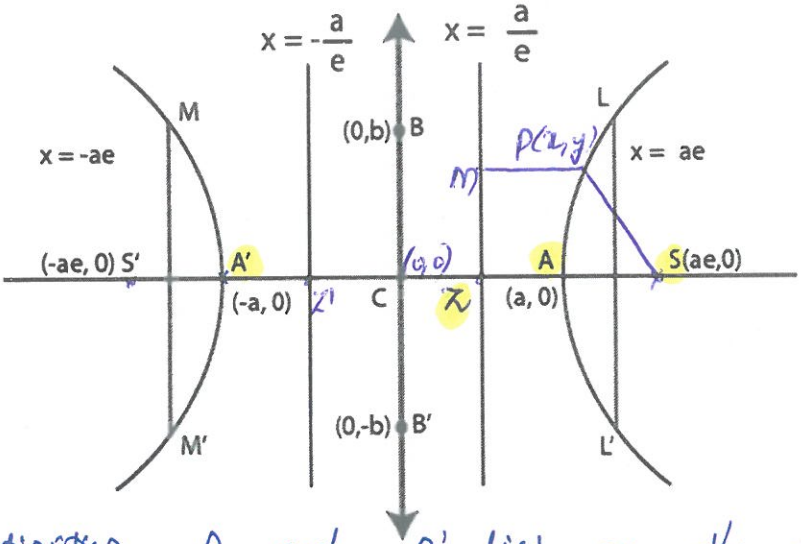
Standard Equation:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The fixed point is called focus, S
the fixed line directrix, ZM
the constant ratio e (>1) eccentricity of hyperbola.

Let us divide SZ internally at A and externally at A' in the ratio e : 1 so that

$$\frac{SA}{AZ} = \frac{e}{1} \quad ; \quad SA = e\,AZ \quad -(1)$$

$$\frac{SA'}{A'Z} = \frac{e}{1} \quad ; \quad SA' = e\,A'Z \quad -(2)$$



Thus, by definition, A and A' lies on the hyperbola.
Let C be the middle point of AA' and AA' = 2a,
then CA = CA' = a.

Adding (1) & (2) $SA + SA' = e\,(AZ + A'Z)$

$$CS - \cancel{CA} + CS + \cancel{CA'} = e\,(A'A)$$

$$\cancel{2}CS = \cancel{2}ae$$

$$CS = ae$$

Subtracting (1) from (2)

$SA' - SA = e(A'Z - AZ)$

$AA' = e[(A'C + CZ) - (CA - CZ)]$

$2a = e\, 2CZ$

$CZ = \frac{a}{e}$ $\quad\therefore$ Equation of ZM $x = \frac{a}{e}$, $x - \frac{a}{e} = 0$

$\therefore$ Co-ordinate of focus $S(ae, 0)$

Equation directrix ZM, $x - \frac{a}{e} = 0$

Using definition of hyperbola, $\frac{PS}{PM} = e\ (>1)$.

We get, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\qquad b^2 = a^2(e^2 - 1)$

length of latus rectum $= \frac{2b^2}{a}$

Note

1) If the centre of the hyperbola is at (h, k) and direction of axes are parallel.

Then eq. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

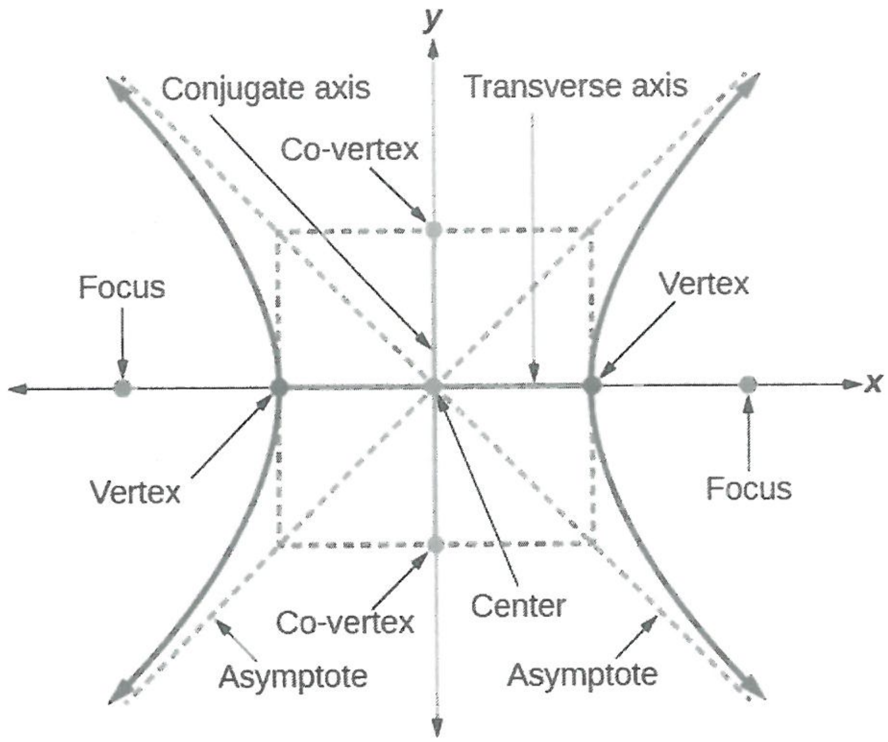
2) Rectangular Hyperbola

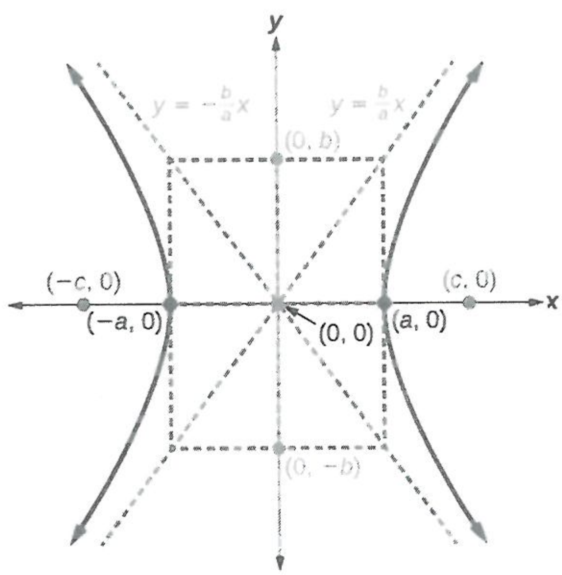
If the lengths of the transverse and conjugate axes are equal, then it is called a Rectangular Hyperbola.

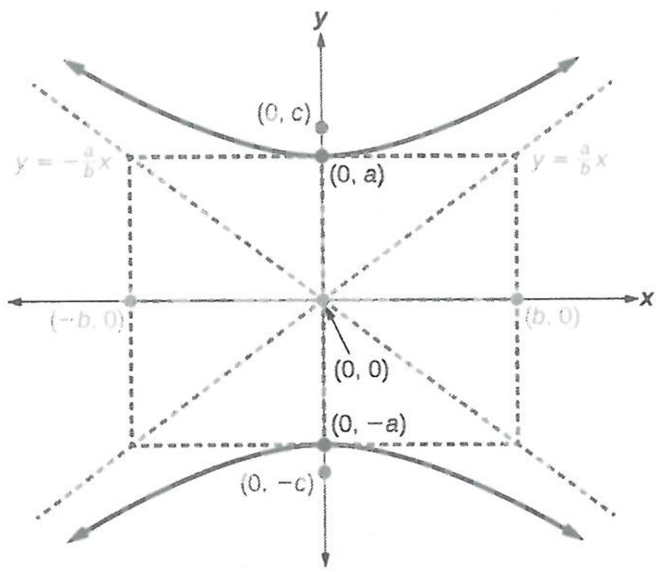
$2a = 2b \Rightarrow a = b \Rightarrow x^2 - y^2 = a^2$

Also, $b^2 = a^2(e^2 - 1) \Rightarrow e = \sqrt{2}$

Eccentricity of rectangular hyperbola is $\sqrt{2}$







Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

Solved Examples

1) Find the equation of the hyperbola whose directrix is $2x+y=1$, focus $(1,2)$ and eccentricity is $\sqrt{3}$.
[$7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$]

2) Find the length of the axes, latus rectum, eccentricity, co-ordinates of the foci and equation of directrices of hyperbola $\frac{x^2}{12} - \frac{y^2}{3} = 1$. [Foci $(\pm\sqrt{15}, 0)$, $x = \pm \frac{4\sqrt{3}}{\sqrt{5}}$
Transverse $= 4\sqrt{3}$, Conjugate axes $= 2\sqrt{3}$, latus rectum $= \sqrt{3}$, $e = \frac{\sqrt{12+3}}{2\sqrt{3}}$]

3) For the hyperbola $\frac{x^2}{100} - \frac{y^2}{25} = 1$, prove that its (i) eccentricity $= \frac{\sqrt{5}}{2}$
(ii) $SA \cdot S'A = 25$, where S & S' are foci and A is the vertex.

4) Find the equation of the hyperbola satisfying the following:
a) Vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$ [$\frac{x^2}{25} - \frac{y^2}{24} = 1$]

(b) Vertices $(0, \pm 7)$, $e = \frac{4}{3}$ [$y^2/49 - \frac{9x^2}{343} = 1$]

(c) Foci $(0, \pm\sqrt{10})$, passing $(2,3)$. [$y^2 - x^2 = 5$]

(4) Find the equation of the hyperbola (referred to its principal axes), if (i) the lengths of transverse and conjugate axes are 5 and 4 respectively. (ii) the length of conjugate axis is 5 and the distance between its foci is 13. (iii) the length of the transverse axis is 7 and it passes through $(5,-2)$ (iv) focus at $(4,0)$ and corresponding directrix is $x=1$.

[(i) $\frac{4x^2}{25} - \frac{y^2}{4} = 1$ (ii) $\frac{x^2}{36} - \frac{4y^2}{25} = 1$ (iii) $\frac{4x^2}{49} - \frac{51y^2}{196} = 1$
(iv) $\frac{x^2}{4} - \frac{y^2}{12} = 1$]